OPTIMAL INVESTMENT

1 . (i) If W is a Brownian motion, prove that for any $a \in \mathbb{R}$ the process $M_t = \exp(aW_t - \frac{1}{2}a^2 t)$ is a martingale, in two ways:

1. Use distributional properties of Brownian motion to calculate $E[M_t \,|\, \mathcal{F}_s]$ for $0 \leq s \leq t$;
2. Use Itô's formula to discover that M is a local martingale, and then establish estimates sufficient to prove that M is in fact a martingale.

(ii) Using (i) or otherwise, prove that $W_t^2 - t$ is a martingale.

(iii) If X is a continuous real-valued process such that for every $a \in \mathbb{R}$ the process $\exp(aX_t - \frac{1}{2}a^2 t)$ is a martingale, prove that X is a Brownian motion.

(iv) If we use the martingale $M_t = \exp(aW_t - \frac{1}{2}a^2 t)$ to define a new probability measure $\tilde{P}$ by the recipe

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = M_t,$$

prove that under $\tilde{P}$ the process $\tilde{W}_t = W_t - at$ is a Brownian motion.

2 . If X is a Brownian motion in $\mathbb{R}^n$, $n > 1$, started at 0, prove that

$$M_t \equiv |X_{1+t}| - \int_0^t \frac{n-1}{2|X_{1+s}|}\, ds$$

is a local martingale. What is its covariation process? In the case $n = 3$, show that

(i) $Y_t \equiv 1/|X_{1+t}|$ is a local martingale;

(ii) Y is bounded in L^2;

(iii) Y is not a martingale.

3 . If X, Y are independent Brownian motions, find the Itô expansions of the following semimartingales:

(i) Y_t/X_t;

(ii) $\tan^{-1}(Y_t/X_t)$;

(iii) $(X_t, Y_t)/\sqrt{X_t^2 + Y_t^2}$;

(iv) $X_t/(X_t^2 + Y_t^2)$.

In each case, comment on any possible issues concerning the application of Itô's formula.

4 . If $Z_t = X_t + iY_t$ is a complex Brownian motion (so X and Y are independent standard Brownian motions), and f is an analytic function, prove that $f(Z)$ is a local martingale. What is its covariation process? Find the Itô expansion of $\log Z_t$.

5 . Suppose that the real-valued process X solves the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_0 \in \mathbb{R},$$

where $\sigma > 0$ and b are Lipschitz functions. Assuming that σ^{-2} is locally integrable, prove that $s(X_t)$ is a local martingale, where

$$s'(x) = \exp\Big(- \int^x \frac{2b(y)}{\sigma(y)^2}\, dy \Big)$$

and deduce that for $a < x_0 < b$, and $\tau = \inf\{t : X_t \notin (a,b)\}$

$$P(X(\tau) = b) = \frac{s(x_0) - s(a)}{s(b) - s(a)}.$$

6 . With standard multivariate asset dynamics

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

and objective

$$E\Big[\int_0^T U(t, c_t)\, dt + U(T, w_T) \Big],$$

where

$$U(t,c) = ae^{-\rho t}u(c) \quad (0 \le t < T), \quad U(T,c) = bu(c)$$

and $u(c) = c^{1-R}/(1-R)$ for some $R > 0$, $R \neq 1$, prove that

$$\begin{aligned} V(t,w) &= f(t)u(w) \\ \pi_t &= \pi_M\, w_t \\ c_t &= \gamma(t)\, w_t \end{aligned}$$

where

$$\begin{aligned} f(t) &= \Big\{ b^{1/R} e^{-q(T-t)/R} + \frac{Ra^{1/R}}{\rho + q} e^{-\rho t/R} \big(1 - e^{-(\rho+q)(T-t)/R} \big) \Big\}^R \\ \pi_M &= R^{-1}(\sigma\sigma^T)^{-1}(\mu - r\mathbf{1}) \\ \gamma(t) &= a^{1/R} e^{-\rho t/R} f(t)^{-1/R} \end{aligned}$$

and

$$q \equiv (R-1)(r + |\kappa|^2/2R), \quad \kappa \equiv \sigma^{-1}(\mu - r\mathbf{1}). \tag{0.1}$$