Yeo D AFM Lec 9

We now have I defined on $\bar{\mathcal{S}}$. But what is $\bar{\mathcal{S}}$?

Claim: $\bar{\mathcal{S}} = \{\text{previsible processes } H \text{ s.t. } E(\int_0^\infty H_s^2 ds) < \infty\}$.

Definition: The previsible σ-field on $\bar{\Omega}$ is the σ-field generated by all left-cts adapted processes.

Remarks: If $Z \in b\mathcal{F}_s$ (bounded), then process $H = Z\, I_{(s,t]}$ is left-cts, adapted as:
$\{H_u > a\} = \{Z > a\} \in \mathcal{F}_s \subseteq \mathcal{F}_u$ for $u \in (s,t]$.

If on the other hand, H is left-cts adapted bdd, we can approximate H by:
$H^{(n)}(t,\omega) = H(2^{-n}\lfloor 2^n t\rfloor, \omega)$. Since $2^{-n}\lfloor 2^n t\rfloor \uparrow t$ $(n \to \infty)$, and H left-cts,
we get $H^{(n)}(t,\omega) \to H(t,\omega)$ as $n \to \infty$ $\forall t, \omega$, and each $H^{(n)}$ is in $\mathcal{S}$.
So any left-cts adapted process H can be approximated by $H^{(n)} \in \mathcal{S}$, so:
$\sigma(\{\text{left-cts adapted processes}\}) = \sigma(\mathcal{S})$ [adapted in limit as $\{H > a\} = \{\lim H^{(n)} > a\} = \bigcup_n \bigcap_{k \geq n} \{H^{(k)} > a\}$.]

Final step is to apply a suitable monotone class theorem.

$f(t,B_t) - f(0,B_0) - \int_0^t \mathcal{L}f(s,B_s)\,ds$ is a m'gale, maybe $= \int_0^t ?\, dB_s$ (limit of). What is the integrand?

Set $t_j^n = (j2^{-n}) \wedge t$, $B_j^n = B(t_j^n)$.

$f(t,B_t) - f(0,B_0) = \sum_{j \geq 1} \{f(t_j^n, B_j^n) - f(t_{j-1}^n, B_{j-1}^n)\}$

$= \sum_{j\geq 1} \{f(t_j^n, B_j^n) - f(t_{j-1}^n, B_j^n) + f(t_{j-1}^n, B_j^n) - f(t_{j-1}^n, B_{j-1}^n)\}$ $(h = 2^{-n})$.

MVT. Don't really care what it is

$= \sum_{j\geq 1} \{h \dot{f}(t_{j-1}^n, B_j^n) + \frac{1}{2}h^2 \ddot{f}(\tilde{t}_{j-1}^n, B_j^n) + (B_j^n - B_{j-1}^n) f'(t_{j-1}^n, B_{j-1}^n)$
$+ \boxed{\frac{1}{2}(B_j^n - B_{j-1}^n)^2 f''(t_{j-1}^n, B_{j-1}^n)} + \frac{1}{6}(B_j^n - B_{j-1}^n)^3 f'''(t_{j-1}^n, \tilde{B}_{j-1}^n)\}$

Examine ~~the~~ the various terms in the expansion. If we restrict attention to $t \leq N$, say, the number of non-zero terms in the sum is $N2^n$.

- The first term is $\sum h \dot{f}(t_{j-1}^n, B_j^n) \to \int_0^t \dot{f}(s,B_s)\,ds$ as it is a Riemann-sum approx to $\int$.
- The second term is bounded by $\sum_{j=1}^{N2^n} \frac{1}{2} h^2 C = \frac{1}{2} C 2^{-2n} \cdot N 2^n \to 0$ (as assume $\ddot{f} \leq C$).
- Final term can be estimated: $E|(B_j^n - B_{j-1}^n)^3 f'''(\sim)| \leq C\, E|B_h|^3 \leq C h^{3/2}$

So $E(|\sum^{N2^n}(B_j^n - B_{j-1}^n)^3 f'''(\sim)|) \leq C \cdot N \cdot 2^n \cdot 2^{-3n/2} = CN2^{-n/2} \to 0$ so fast that by BC1, $|\sum(B_j^n - B_{j-1}^n)^3 f'''(\sim)| \to 0$ a.s.

- For the stochastic integral term:

$\sum (B_j^n - B_j^{n-1}) f'(t_{j-1}^n, B_{j-1}^n) = \int_0^t f'(2^{-n}[\triangle 2^n], B(2^{-n}[\triangle 2^n]))\,dB_s$.

But we know: $f'(2^{-n}[\triangle 2^n], B(2^{-n}[\triangle 2^n])) \to f'(s,B_s)$ uniformly

So the stochastic integrals converge to $\int_0^t f'(s,B_s)\,dB_s$.

- For the second derivative $\boxed{}$ term:

$= \sum_{j\geq 1} \frac{1}{2}\{(B_j^n - B_{j-1}^n)^2 - h\} f''(t_{j-1}^n, B_{j-1}^n) + \underbrace{\sum_{j\geq 1} h \cdot \frac{1}{2} f''(t_{j-1}^n, B_{j-1}^n)}_{\to \int_0^t \frac{1}{2} f''(s,B_s)\,ds}$

Now, let's write $X_j = (B_j^n - B_{j-1}^n)^2 - h$, which is a zero-mean RV, with variance $= ch^2$

Consider $E[(\sum_j \xi_j X_j)^2]$ $\quad (\xi_j \equiv \frac{1}{2} f''(t_{j-1}^n, B_{j-1}^n)$, is $\mathcal{F}_{t_{j-1}^n}$-meas.$)$

$= E[\sum \xi_j^2 X_j^2 + 2\sum_{j<k} \xi_j \xi_k X_j X_k]$ $\boxed{*}$

X_js are squared increments of BM over disjoint intervals (so independent etc.)

Note $\xi_j \xi_k X_j$ known at time t_{k-1}^n, & $E[X_k | \mathcal{F}_{t_{k-1}^n}] = 0 \leftarrow X_k$s are 0-mean.

So off-diagonal terms at $\boxed{*}$ disappear.

$\therefore = E[\sum \xi_j^2 X_j^2] \leq C \cdot N 2^{-n} \cdot c \cdot 2^{-2n} \longrightarrow 0$.

Conclusion: $f(t,B_t) - f(0,B_0) - \int_0^t (\mathcal{L}f)(s,B_s)\,ds = \int_0^t f'(s,B_s)\,dB_s$

$df(t,B_t) = f'(t,B_t)\,dB_t + (\mathcal{L}f)(t,B_t)\,dt$
$= f'(t,B_t)\,dB_t + \dot{f}(t,B_t)\,dt + \frac{1}{2} f''(t,B_t)\,dt$