AFM 24th Oct (Mon)

Recall $f: [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is $C^{1,2}$, satisfies exponential growth condition, then

$$f(t, B_t) - f(0, B_0) - \int_0^t \mathcal{L} f(s, B_s)\, ds \quad \text{is a martingale.}$$

Eg $f(t,x) = x^2 - t$ or $f(x,t) = \exp(\theta x - \frac{1}{2}\theta^2 t) \Rightarrow \mathcal{L} f = 0$

$[\mathcal{F}_t^\circ = \sigma(\{B_s ; s \le t\})$; $\mathcal{F}_t = (\bigcap_{s>t} \mathcal{F}_s^\circ) \vee \mathcal{N}$ ← all null sets

$M_t \quad \mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ Want M to have right conti paths.

$\tilde{M}_t = \liminf_{s \downarrow t,\, s \in \mathbb{Q}} M_s$

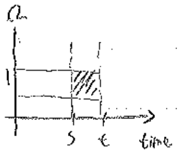
<u>Basics on stochastic integration</u>

We wan to define for H in some suitable class of processes

$$I(H) = \int_0^\infty H_s\, dB_s = (H \cdot B)$$

As with all integration theories, we begin with "elementary" integrations, extend by linearity to "simple" integrands, and then push the theory all the way by some approximation story.

The elementary integrands here are all processes of the form



$$H = I_A I_{(s,t]} \qquad [H(u,\omega) = I_A(\omega) \cdot I_{(s,t]}(u)]$$

for any $0 \le s \le t$, any $A \in \mathcal{F}_s$. We think of H as a map from $\bar{\Omega} = (0,\infty) \times \Omega$

$$\int_0^\infty H_u \cdot dB_u = I_A (B_t - B_s)$$

This is the definition of $I(H)$ for these elementary H.

Notice that $I(H)$ is a random variable which is in fact square integrable;

$$\begin{aligned}
\mathbb{E}[(H \cdot B)^2] &= \mathbb{E}[I_A^2 (B_t - B_s)^2] = \mathbb{E}[I_A (B_t - B_s)^2] \\
&= \mathbb{E}[I_A (t-s)] \qquad (\because B_t^2 - t \text{ mart}, \ A \in \mathcal{F}_s) \\
&= \mathbb{E}\left[\int_0^\infty H_u^2\, du\right]
\end{aligned}$$

So the first extension we make is to integrands of the form

$$H = Z \cdot I_{(s,t]}$$

where Z is bdd $\mathcal{F}_s$-msble, $(\Rightarrow N \cdot P)$

$$I(H) = Z(B_t - B_s)$$

Again, $\mathbb{E}[(H\cdot B)^2] = \mathbb{E}[\int_0^\infty H_u^2\,du]$

The next extension is to integrands of the form

$$H = \Sigma_{j=1}^N Z_j\, I_{(s_j,\, t_j]} \qquad (*)$$

where Z_j is bdd, $\mathcal{F}_{s_j}$-msble, we will have to have

$$I(H) = \Sigma_{j=1}^N Z_j (B_{t_j} - B_{s_j})$$

It needs to be checked that this answer is indep of the way we represent H as a sum, but it's OK.

We can again check for such simple integrands that

$$\mathbb{E}[(H\cdot B)^2] = \mathbb{E}[\int_0^\infty H_u^2\,du]$$

This is a little more involved, but if we suppose $s_1 \le t_1 \le s_2 \le t_2 \cdots \le s_N \le t_N$ then we have

$$\mathbb{E}[(\sum_{j=1}^N Z_j(B_{t_j} - B_{s_j}))^2] = \mathbb{E}[\sum_{j=1}^N Z_j^2 (B_{t_j} - B_{s_j})^2 + 2\sum_{j<k} \underbrace{Z_j Z_k (B_{t_j} - B_{s_j})}_{\text{all in } \mathcal{F}_{s_k}} (B_{t_k} - B_{s_k})]$$

$$= \mathbb{E}[\sum_j Z_j^2 (B_{t_j} - B_{s_j})^2] \qquad \left\{ \mathbb{E}[G\cdot(B_{t_k} - B_{s_k})] = 0 \text{ if } G \in \mathcal{F}_{s_k} \right\}$$

$$= \mathbb{E}[\sum_j Z_j^2 (t_j - s_j)]$$

$$= \mathbb{E}[\int_0^\infty H_u^2\,du]$$

So if we let $\mathcal{S}$ denote the collection of all simple processes $(*)$, we've defined $I : \mathcal{S} \to L^2(\mathcal{F}_\infty)$, with the following property; if we define a norm $\|\cdot\|_B$ on $\mathcal{S}$ by

$$\|H\|_B = (\mathbb{E}[\int_0^\infty H_s^2\,ds])^{1/2}$$

(in fact, this is just the L^2 norm on $(\Omega, \mathcal{B}(0,\infty)\times\mathcal{F}_\infty, \text{Leb}\times P)$) then the map $I : \bar{\mathcal{S}} \to L^2(\mathcal{F}_\infty)$ is an <u>isometry</u>.

We now exploit this to build the stochastic integral for all $H \in \bar{\mathcal{S}}$ for any such H, there are $H^n \to H$ in $\|\cdot\|_B$ for which we know $I(H^n)$, which form a Cauchy sequence in $L^2(\mathcal{F}_\infty)$; this converges to some limit Y, and we define $I(H) = Y$.