21/10/12

<u>correction</u> - from last time

FTAP - Single time period

$$\inf \mathbb{E} \exp\{-\theta\cdot(S_1 - S_0)\}$$ attained $\Rightarrow \exists$ EMM

not attained $\Rightarrow \exists$ arbitrage

We still need to show that we can't have both $\exists$ EMM and $\exists$ arbitrage. Suppose the contrary, that $\mathbb{Q}$ is an EMM, and θ is an arbitrage. Then $\theta\cdot(S_1 - S_0) \geq 0$

and $\mathbb{P}(\theta(S_1 - S_0) > 0) > 0$; so $\mathbb{Q}(\theta\cdot(S_1 - S_0) > 0) > 0$

Now we have

$$\mathbb{E}^{\mathbb{Q}}(\theta\cdot(S_1 - S_0)) = 0$$ because S is a $\mathbb{Q}$ martingale

$\Rightarrow\Leftarrow$

Remarks from earlier

(i) B.M. exists

(ii) $\forall c \neq 0$ $\quad cB(t/c^2)$ is a Brownian motion

(iii) the process $\tilde{B}_t \equiv \begin{cases} tB(1/t) & t>0 \\ 0 & t=0 \end{cases}$ is a B.M.

(iv) If $\mathcal{F}_t^0 = \sigma(\{B_u : u \leq t\})$, and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^0$, then for any a.s. finite $(\mathcal{F}_t)$-stopping time T, the process

$B_{t+T} - B_T$ is a B.M, independent of $\mathcal{F}_T$

(v) $\mathbb{P}(\sup_t B_t = +\infty, \inf_t B_t = -\infty) = 1$

- So no last visit to 0...

and infact the Brownian motion is with probability 1 no where differentiable

(vi) The process B is a zero mean ~~process~~ Gaussian Process with

$$\mathbb{E}(B_s B_t) = s \wedge t$$

(vii) Easy to see that $B_t^2 - t$ is a martingale

- but the converse is also true; if M is a continuous martingale $M^2 - t$ is a martingale, then M is a B.M. (Levy)

(viii) $\forall a \in \mathbb{R}$, $\exp(a B_t - \frac{1}{2} a^2 t)$ is a martingale

(ix) If $H_a = \inf\{t : B_t = a\}$, then the density of H_a is

$$\frac{\mathbb{P}(H_a \in dt)}{dt} = \frac{a e^{-a^2/2t}}{\sqrt{2\pi t^3}}$$

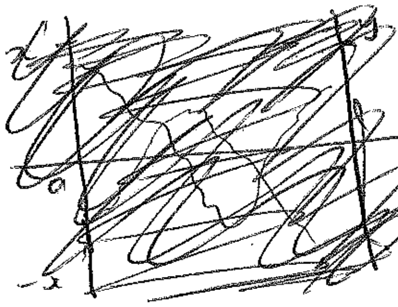
(x) Write $P_t(x,y) \equiv \exp(-\frac{1}{2}(y-x)^2/t) / \sqrt{2\pi t}$

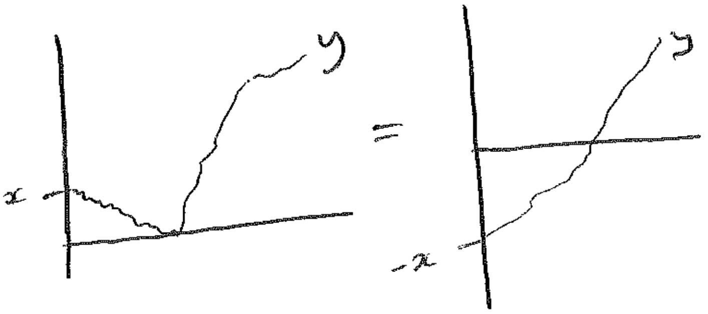
the transition density of the B.M. ($P_t(x,y)\,dy = \mathbb{P}(B_t \in dy \mid B_0 = x)$ ... we'll often call $B_t + at$ a Brownian motion when $a \neq 0$.)

We have by the reflection principle for $x, y > 0$

$$\mathbb{P}(B_t \in dy,\ B_s > 0 \text{ for } s < t \mid B_0 = x) = \{P_t(x,y) - P_t(-x,y)\}\,dy$$

in probability





$\to$ $\boxed{* \; B_t^2 - t}$ is a martingale

$* \; M_t^2 - [M]_t$

Theorem

Suppose $f \; [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is $C^{1,2}$ satisfies an exponential growth bound; for some $k>0$

$$\sup_{t,x} e^{-k|x|} \left[|f(t,x)| + \left|\frac{\partial f}{\partial t}(t,x)\right| + \sum \left|\frac{\partial f}{\partial x_i}(t,x)\right| + \sum_{ij} \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)\right| \right] < \infty$$

Define

$$(\mathcal{L}f)(t,x) = \frac{\partial f}{\partial t} + \frac{1}{2}\Delta f$$

Then

$$\mathbb{E}^x f(t, B_t) = f(0,x) + \mathbb{E}^x \left[\int_0^t \mathcal{L}f(s, B_s)\, ds \right]$$

(Notation $\mathbb{E}^x(\cdot) = \mathbb{E}(\cdot \mid B_0 = x)$)

Remark if $\mathcal{L}f = 0$, then we have $f(t, B_t)$ is a mart.

- you check that $f(t,x) \equiv |x|^2 - dt$ satisfies $\mathcal{L}f = 0$

- $f(t,x) = \exp\left(a.x - \frac{1}{2}|a|^2 t\right)$

Proof We have to observe that (just do $d=1$ for simplicity)

$$\boxed{-\frac{\partial}{\partial t} P_t(x,y) + \frac{1}{2}\frac{\partial^2}{\partial x^2} P_t(x,y) = 0}$$

we'll consider for some $\varepsilon > 0$ (fubini)

$$\mathbb{E}^x\left[\int_\varepsilon^t \mathcal{L}f(s,B_s)\,ds\right] = \int_\varepsilon^t \mathbb{E}^x\left[\mathcal{L}f(s,B_s)\right] ds$$

$$= \int_\varepsilon^t \left\{ \int \mathbb{P}_s(x,y)\,\mathcal{L}f(s,y)\,dy \right\} ds$$

$$= \int_\varepsilon^t \left\{ \int \mathbb{P}_s(x,y) \left(\frac{\partial f}{\partial s} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \right)(s,y)\,dy \right\} ds$$

$$= \int_\varepsilon^t \left\{ \int \left(\mathbb{P}_s(x,y)\frac{\partial f}{\partial s} + \frac{1}{2}\frac{\partial^2 \mathbb{P}_s}{\partial y^2} f(s,y) \right) dy \right\} ds$$

integration by parts

truncation @ ε

$$= \int_\varepsilon^t \left(\int \left\{ \mathbb{P}_s(x,y)\frac{\partial f}{\partial s} + f(s,y)\frac{\partial \mathbb{P}_s}{\partial s}(x,y) \right\} dy \right) ds$$

$$= \int \int_\varepsilon^t \frac{\partial}{\partial s}\left(\mathbb{P}_s(x,y)\, f(s,y) \right) ds\, dy$$

$$= \int \left\{ \mathbb{P}_t(x,y)\, f(t,y) - \mathbb{P}_\varepsilon(x,y)\, f(\varepsilon,y) \right\} dy$$

$$= \mathbb{E}^x\left[f(t,B_t) \right] - \mathbb{E}^x\left[f(\varepsilon, B_\varepsilon) \right]$$

Now we let $\varepsilon \downarrow 0$: use dominated convergence and we get

$$\mathbb{E}^x\left[f(t,B_t)\right] - f(0,x) = \mathbb{E}^x\left[\int_0^t (\mathcal{L}f)(s,B_s)\,ds \right]$$

<u>corollary</u> for any f satisfying the exponential growth ~~coated~~ condition and also C^{12} property,

$$f(t,B_t) - \int_0^t (\mathcal{L}f)(s,B_s)\,ds \quad \text{is a martingale}$$

$$= f(0, B_0) + \int_0^t f'(s, B_s)\, dB_s$$

$$\frac{\partial f}{\partial x}$$

$$\dot{f} = \frac{\partial f}{\partial t} \; ; \; f' = \frac{\partial f}{\partial x}$$