Thm: Suppose $S_t^0 = 1$ for all t. Then t.f.a.e for some X

(i) $X = (\theta \cdot S)_T$ for some previsible portfolio process θ

(ii) $\mathbb{E}^Q X = 0$ for all $Q \in \mathcal{Q}$, s.t $\mathbb{E}^Q |X| < \infty$

Proof: done (i) $\Rightarrow$ (ii)

Now (ii) $\Rightarrow$ (i). To do this, let's write

$$Z_\lambda = \exp\left(-\frac{1}{2}\sum_{t=1}^{T} |\Delta S_t|^2 - \frac{1}{2}|X|^2 - \lambda X\right)$$

for $\lambda = 0, 1$

Now we proceed through the analysis of the FTAP and deduce that there exists a portfolio process $\theta^*(\lambda)$ s.t

$$\frac{dQ_\lambda}{dP} = c_\lambda Z_\lambda \cdot \exp\{-(\theta^*(\lambda)\cdot S)_T\}$$

is an equivalent martingale measure. Then we have

$$\frac{dQ_0}{dP} = C_0 Z_0 \exp\{-(\theta^*(0)\cdot S)_T\}$$

$$\frac{dQ_1}{dP} = C_1 Z_0 \exp\{-(\theta^*(1)\cdot S)_T - X\}$$

Hence we have

$$X = -\log(C_0/C_1) + ((\theta^*(0) - \theta^*(1))\cdot S)_T + \log\left(\frac{dQ_0}{dQ_1}\right)$$

So we calculate

$$
\begin{aligned}
0 = \mathbb{E}^{Q_1}(X) &= -\log(C_0/C_1) = -\log(C_0/C_1) + \mathbb{E}^{Q_1}\log\left(\frac{dQ_0}{dQ_1}\right) \\
&\le -\log(C_0/C_1) + \log\left(\mathbb{E}^{Q_1}\frac{dQ_0}{dQ_1}\right) \\
&= -\log(C_0/C_1) \\
&= -\log(C_0/C_1) - \log\left(\mathbb{E}^{Q_0}\left(\frac{dQ_1}{dQ_0}\right)\right) \\
&\le -\log(C_0/C_1) - \mathbb{E}^{Q_0}\left(\log\left(\frac{dQ_1}{dQ_0}\right)\right) \\
&= \mathbb{E}^{Q_0}(X) = 0
\end{aligned}
$$

So everything is zero; $C_0 = C_1$ and $\frac{dQ_1}{dQ_0} = 1$ a.s

Remark: If there is only one EMM, then for any $X \in L^1(Q)$, X can be represented as

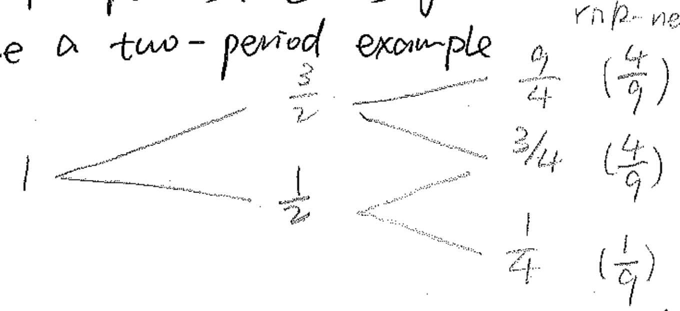
$$X = \mathbb{E}^Q X + (\theta \cdot S)_T$$

for some previsible portfolio process θ.

Binomial pricing

We've already seen how this works. The methodology extends to multiple periods. Let's just work through an example.

Take a two-period example



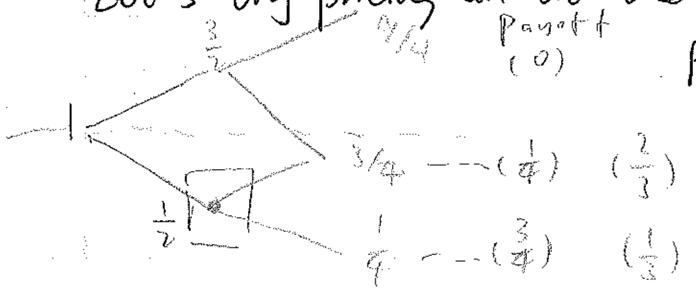
For the bank account, $r = \frac{1}{6}$

Calculate the pricing probability: If p is the prob of a move of stock to $\frac{3}{2}$, then we need

$$p\frac{3}{2} + (1-p)\frac{1}{2} = 1 + \frac{1}{6} = \frac{7}{6}$$

$$\Rightarrow \quad p = \frac{7}{6} - \frac{1}{2} = \frac{7}{6} - \frac{3}{6} = \frac{2}{3} \qquad \left(\mathbb{E}^{Q}\left(\frac{S_1}{1+r}\right) = S_0 \right)$$

Let's try pricing an at-the-money European put with expiry $T=2$.



Payoff at time 2 is

$(K - S_2)^+ = (1 - S_2)^+$

So the time-0 price of the put option is

$$\left(\frac{6}{7}\right)^2 \left\{ \frac{4}{9} \times \frac{1}{4} + \frac{1}{9} \times \frac{3}{4} \right\} = \left(\frac{6}{7}\right)^2 \cdot \frac{7}{36} = \frac{1}{7}$$

How about an <u>American</u> put option? This's an option where you can choose to exercise at any stopping time $\leq T$

The only time in this example ~~that~~ when we have to think would be at time 1. If $S_1 = \frac{1}{2}$.

~~If $S_1 = \frac{1}{2}$~~. If we're there and we continue, the value of our reward at time 2 will be

$$\frac{6}{7} \times \left\{ \frac{2}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{3}{4} \right\} = \frac{6}{7} \times \frac{5}{12} = \frac{5}{14}$$

If we stop at that node, we get $(K - S_1)^+ = \frac{1}{2} > \frac{5}{14}$

So our optimal choice would be to stop at that node

What's the early-exercise premium?

American is worth $\frac{1}{2} - \frac{5}{14} = \frac{1}{7}$ more than European at the node $S_1 = \frac{1}{2}$, we can calculate

$$\frac{6}{7} \times \frac{1}{3} \times \frac{1}{7} = 2/49$$

Brownian motion and the Black-Scholes model

Def: A Brownian motion $(B_t)_{t \in [0,\infty)}$ is a stochastic process s.t

(i) $B_0 = 0$

(ii) $t \mapsto B_t(w)$ is continuous for all w

(iii) for all $0 \le s \le t$,

$$B_t - B_s \sim N(0, t-s) \text{ independent of } \sigma(B_u : u \le s)$$

Remark: i) Brownian motion exits (non-trivial)