AFM(4) 14/10/2011

Cor If $S_t^0 > 0$ $\forall t$, then the th^m applies to $\tilde{S}_t = S_t / S_t^0$

We saw the example of $S_t^0 = (1+r)^t$ & S^1 is a stock S^2 is a European call option paying $(S_T^1 - K)^+$ at time T.

By FTAP, $\hat{S}_t^2 = S_t^2 / S_t^0 = S_t^2/(1+r)^t$ is a Q-martingale. so $\hat{S}_t^2 = \mathbb{E}^Q(\hat{S}_T^2 | \mathcal{F}_t)$,

$$= \frac{S_t^2}{(1+r)^t} = \mathbb{E}^Q\left(\frac{S_T^2}{(1+r)^T} \Big| \mathcal{F}_t\right)$$

Now notice that $Q \ll P$ on each $\mathcal{F}_t$, so $\exists$ an $\mathcal{F}_t$-measurable density $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = M_t$, which is easily seen to be a P-martingale:

because $\forall A \in \mathcal{F}_t$ $\int_A M_t \, dP = \int_A \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} dP = Q(A)$

$$= \int_A \left.\frac{dQ}{dP}\right|_{\mathcal{F}_{t+1}} dP \quad \text{since } A \in \mathcal{F}_{t+1}$$

$$= \int_A M_{t+1} \, dP.$$

We can likewise easily show that for $t \leq n$; $Z \in \mathcal{F}_t$ $\mathbb{E}^Q(Z | \mathcal{F}_t) = \mathbb{E}^P(Z M_n | \mathcal{F}_t) / M_t$.

$$\frac{S_t^2}{(1+r)^t} = \mathbb{E}^P\left(\frac{S_T^2}{(1+r)^T} M_T \Big| \mathcal{F}_t\right) \Big/ M_t.$$

So if we define $\boxed{\zeta_t = M_t (1+r)^{-t}}$

we get $\boxed{S_t^2 = \frac{1}{\zeta_t} \mathbb{E}(\zeta_t S_T^2 | \mathcal{F}_t)}$

LDP.T.O

Pf (FTAP)

Let just do the case $T=1$; we'll suppose that $S_t^0 = 1 \ \forall t$; and we'll suppose that the problem is non-degenerate in the sense that

$$\mathbb{P}(\theta \cdot (S_1 - S_0) = 0) < 1. \quad \forall \theta \in \mathbb{R}^N, \theta \neq 0.$$

This is an innocent assumption: if it fails, some assets are linearly dependent on each other, so we could discard the redundant assets.

(Sketch of the idea)

$$\sup_\theta \mathbb{E}\left(-\exp(-\theta \cdot (S_1 - S_0))\right) \qquad U(x) = -e^{-x}$$

If we can find θ^* which attain sup., then differentiate w.r.t θ.

$$0 = \mathbb{E}\left[(S_1 - S_0) \underbrace{-\exp(-\theta^* \cdot (S_1 - S_0))}_{\propto \text{ density of EMM}}\right].$$

To begin with, we clear the ground we shall define

$$\theta \mapsto \varphi(\theta) = \mathbb{E}\left[\exp\{-\theta \cdot (S_1 - S_0) - \tfrac{1}{2}|S_1 - S_0|^2 - \lambda \xi - \tfrac{1}{2}\xi^2\}\right]$$

where ξ could be any r.v. λ any real.
We need this for 2nd FTAP, but for now, we can think that $\xi = 0$

$$\mathbb{E}\left[(S_0 - S_1) e^{-\theta^* (S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2}\right]$$

Pf (ctd)

The f^n $\varphi : \mathbb{R}^N \to \mathbb{R}$ is bounded below, so we consider the problem $\inf_{\theta \in \mathbb{R}^N} \varphi(\theta)$.

2 possibilities arise: ① The inf is attained at θ^*. In that case the (differentiable) f^n. φ has a vanishing derivative at θ^* $\quad 0 = D\varphi(\theta^*) = \mathbb{E}\left[(S_0 - S_1) e^{-\theta^*(S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2}\right]$

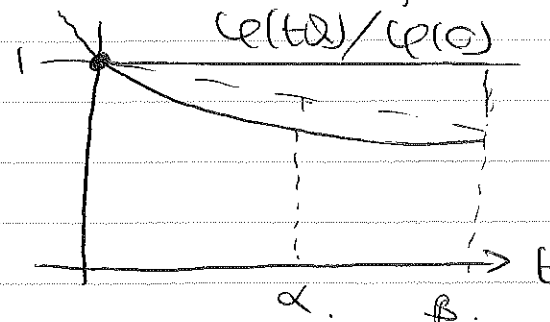
If we define $\dfrac{dQ}{dP} = \dfrac{e^{-\theta^*(S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2}}{\mathbb{E}\left(e^{-\theta^* \cdot (S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2}\right)}$

Then $\mathbb{E}^Q(S_1 - S_0) = 0$, that is, $S_0 = \mathbb{E}^Q(S_1)$
that is S_1 is a Q-martingale.

② The infimum is <u>not</u> attained, so there exist (Q_n), $|Q_n| \to \infty$, s.t. $\varphi(Q_n) \downarrow \inf \varphi(Q)$.
Let's consider for $\alpha > 0$ the set F_α defined by
$F_\alpha = \left\{ Q : |Q| = 1 \right\}$ s.t. $\dfrac{\varphi(\alpha Q)}{\varphi(0)} \le 1 \Big\}$

This is a closed subset of S^N

We also see, from the convexity of φ, that for $0 < \alpha < \beta$, we must have $F_\beta \subseteq F_\alpha$.



Then we use the finite intersection property (FIP) of the closed subsets $\{F_\alpha\}$ of the compact set S^{N-1}
if $\bigcap_{\alpha > 0} F_\alpha = \emptyset$, then $F_a = \emptyset$ for a

But if $F_a = \emptyset$ then $\varphi(aQ)/\varphi(0) > 1$ $\forall Q \in S^{N-1}$. Thus $\inf_{|Q| \geq a} \varphi(Q)/\varphi(0) > 1$.
by convexity of φ. Then the inf of $\varphi(Q)$ is attained. ∴ (ctd) #.

So it must be that $\bigcap_\alpha F_\alpha \neq \emptyset$. This means there is some $v \in S^{N-1}$ st $\forall t \geq 0$.
$\varphi(tv) \le \varphi(0)$. That is, $\mathbb{E}[\exp(-tv(S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2)]$
$\le \mathbb{E}[\exp(-\frac{1}{2}|S_1 - S_0|^2)]$.
∴ $\mathbb{P}[v.(S_1 - S_0) < 0] = 0$, o/w as $t \to \infty$ L.H.S gets unboundedly large. But $\mathbb{P}(v.(S_1 - S_0) = 0) < 1$
so $\mathbb{P}(v.(S_1 - S_0) \geq 0) = 1$, $\mathbb{P}(v.(S_1 - S_0) > 0) > 0$.
ie. an arbitrage. □

AFM(5) 17/10/2011

FTAP: $\min_{\theta} \mathbb{E}[\exp(-\theta . S) . e^{-\frac{1}{2}|S_1|^2}]$

$S^0_t = 1, \ t = 0, 1$

The next stage is to extend to multiperiod settings. We'll use the notation $(\theta . S)_t$ for the gains from trade of portfolio process θ by time t;

This means $(\theta . S)_t = \sum_{j=1}^{t} \theta_j . \Delta S_j$ ∵ there might be a cash acc.

where $\Delta S_j = S_j - S_{j-1}$. Notice that $(\theta . S)_t \neq \theta_t . S_t$

But we do have that $\theta_t . S_t = \theta_0 . S_0 + (\theta . S)_t$.

The ext^n to multiperiod is proved under these assumptions $S^0_t = 1$. Then the result is this:

(FTAP) TFAE: (i) There is no arbitrage.

(ii) There exists $Q \sim P$ s.t. (S_t) is a Q martingale.

Pf: (ii) $\Rightarrow$ (i) If θ were an arbitrage, then

$(\theta_0 . S_0) \leq 0 \leq (\theta_T . S_T)$; $(\theta . S)_T \geq 0$ &

$\mathbb{P}((\theta . S)_T > 0) > 0$

In that case $Q((\theta . S)_T > 0) > 0$.

But $\mathbb{E}^Q((\theta . S)_T) = \mathbb{E}^Q(\sum_{j=1}^{T} \theta_j . \Delta S_j)$

$= \mathbb{E}^Q[\sum_{j=1}^{T} \mathbb{E}^Q(\underline{\theta_j} . \Delta S_j \mid \mathcal{F}_{j-1})]$

$\mathcal{F}_{j-1}$ - measurable

$= 0$ ∵ under Q S_j is a martingale

(i) $\Rightarrow$ (ii) We shall construct a portfolio process θ^* by backwards induction, using the one-step argument. Let's define

$$Z_k = \exp\left(-\frac{1}{2}\sum_{j=1}^{T} |\Delta S_j|^2 - \sum_{j=k+1}^{T} \theta^*_j . \Delta S_j\right)$$

with $Z_T = \exp\left(-\frac{1}{2}\sum_{j=1}^{T} |\Delta S_j|^2\right)$.

Pf (ctd): Now start at the end & consider the problem
$$\inf_{\theta \in \mathbb{R}^N} \mathbb{E}\left[\exp(-\theta.\Delta S_T) Z_T \mid \mathcal{F}_{T-1}\right]$$
Since there is no arbitrage there must be some θ_T^* which achieves the inf, and as we saw in the one-step proof
$$\mathbb{E}_{T-1}\left(\Delta S_T . \exp(-\theta_T^*.\Delta S_T) Z_T\right) = 0 .$$
Notice that θ_T^* must be a random variable: we have a technical point to show that θ_T^* can be chosen $\mathcal{F}_{T-1}$ - measurable. This begins the construction of the EMM, which will in fact be given by Z_0. Notice that as we add further factors to Z, we get

Look up!

$$\mathbb{E}_{T-1}\Big[\Delta S_T \exp(-\theta_T^*.\Delta S_T) . \underbrace{Z_T \exp\Big(-\sum_{k=t+1}^{T} \theta_j^* . \Delta S_j\Big)}_{= Z_k}\Big]$$
$$= \mathbb{E}_{T-1}\left[\left(\Delta S_T \exp(-\theta_T^*.\Delta S_T) Z_T\right)\right] \exp\Big(-\sum_{j=t+1}^{T} \theta_j^* \Delta S_j\Big)$$
$$= 0 .$$
That is the martingale property at the final time step, is not affected as we go back.
The induction is repeated use of the one-step argument:
$$\inf_{\theta} \mathbb{E}\left[(\exp(-\theta.\Delta S_k)) Z_k \mid \mathcal{F}_{k-1}\right]$$
is the problem which is attained at some $\mathcal{F}_{k-1}$ - measurable θ_k^*. $\square$

To finish off, if we have a numeraire asset $N_t \equiv S_t^0 > 0$, then we can apply the FTAP to $\tilde{S}_t \equiv S_t / S_t^0$: no arbitrage iff $\exists Q \sim P$ s.t. $\tilde{S}$ is a Q - martingale. So in that case
$$\tilde{S}_t = S_t / S_t^0 = \mathbb{E}^Q\left(S_T / S_T^0 \mid \mathcal{F}_t\right) = \frac{1}{\lambda_t} \mathbb{E}^P\left(\lambda_t S_T / S_T^0 \mid \mathcal{F}_t\right)$$

P.T.O.

where $\lambda_t = \frac{dQ}{dP}\Big|_{\mathcal{F}_t}$

Thus if $\zeta_t = \lambda_t S_t^0$, we get $\boxed{}$ again:

$$\boxed{S_t = \frac{1}{\zeta_t}\mathbb{E}^P(\zeta_T S_T|\mathcal{F}_t)}$$

<u>The Second Fundamental Theorem of Asset Pricing</u>

Let $\mathcal{Q}$ denote the set of all equivalent martingale measures ($S_t^0 \equiv 1$ assumed) Then we have the following statment/result

<u>Th^m</u>: ~~...~~ Suppose X is an $\mathcal{F}_T$-measurable r.v. TFAE

(i) For some previsible process θ s.t. $X = (\theta.S)_T$

(ii) $\forall Q \in \mathcal{Q}$ s.t. $\mathbb{E}^Q(|X|) < \infty$, $\mathbb{E}^Q(X) = 0$.

<u>Remark</u> A very important case is when $\mathcal{Q} = \{Q\}$, a singleton. Then the th^m says that a r.v. X is replicable $\iff \mathbb{E}^Q(X) = 0$. This is called a <u>complete</u> market.

<u>Pf</u>: (Tehranchi)

<u>(i) $\Rightarrow$ (ii)</u>: We have $X = (\theta.S)_T$, and $Q \in \mathcal{Q}$

The thing we need to check is that if we condition X back it remains a martingale. Suppose we have $X_t \equiv (\theta.S)_t$; we know $X_T \equiv X \in L^1(Q)$

Then we will prove by induction that $X_t \in L^1(Q)\ \forall t$

To do that, suppose we know $X_t \in L^1(Q)$ and we consider ~~$A_n = \{|X_t| \le n, |\theta_{t-1}| \le n$~~

$$A_n = \{|X_{t-1}| \le n;\ |\theta_t| \le n\} \in \mathcal{F}_{t-1}$$

Then $\mathbb{E}[X_t \mathbb{I}_{A_n}|\mathcal{F}_{t-1}] = \mathbb{E}[(X_{t-1} + \theta_t . \Delta S_t)\mathbb{I}_{A_n}|\mathcal{F}_{t-1}]$

$$= X_{t-1}\mathbb{I}_{A_n} + \mathbb{I}_{A_n}\theta_t \cdot \underbrace{\mathbb{E}^Q(\Delta S_t|\mathcal{F}_{t-1})}_{=0 \ \because S \text{ a } Q\text{-mart}}$$

But $X_t \in L^1(Q)$, so conditional dominated convergence can be applied to L.H.S to give us $\mathbb{E}^Q(X_t|\mathcal{F}_{t-1}) = X_{t-1}$

Thus $|X_{t-1}| \le \mathbb{E}^Q(|X_t||\mathcal{F}_{t-1}) \in L^1(Q)$

Thus $X_t - X_{t-1} = \theta . \Delta S_t$ is integrable.

& $\mathbb{E}^{Q}(X_t - X_{t-1}) = 0$

since S is a Q-martingale

θ_t is $\mathcal{F}_{t-1}$-measurable