e $\mathbb{Q} \sim \mathbb{P}$ means $\mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$ for any event $A$.
e or: there exists a strictly positive $\mathbb{P}$-integrable $Z$ s.t. $\mathbb{Q}(A) = \int_A Z \, d\mathbb{P}$, by

Radon-Nikodym Theorem.
, for example, if $S_t^1$ is a stock and $S_t^2$ an option on it then $S_t^2 = \mathbb{E}^{\mathbb{Q}}[S_T^2 | \mathcal{F}_t]$
$= \mathbb{E}^{\mathbb{Q}}[(S_T^1 - K)^+ | \mathcal{F}_t]$

arks Let us suppose there is some positive (strictly) adapted process $(N_t)_{0 \le t \le T}$
eaning $N_t$ is $\mathcal{F}_t$-measurable) and we define

$$\tilde{S}_t^i = \frac{S_t^i}{N_t}$$

$$= \frac{W_t}{N_t}. \text{ Then } \tilde{W}_{t+1} - \tilde{W}_t = \frac{W_{t+1}}{N_{t+1}} - \frac{W_t}{N_t} = \frac{\bar{\Theta}_{t+1} \cdot \bar{S}_{t+1}}{N_{t+1}} - \frac{\bar{\Theta}_t \cdot \bar{S}_t}{N_t}$$

$$= \bar{\Theta}_{t+1} \cdot \left( \frac{\bar{S}_{t+1}}{N_{t+1}} - \frac{\bar{S}_t}{N_t} \right) = \bar{\Theta}_{t+1} \cdot \left( \tilde{S}_{t+1} - \tilde{S}_t \right)$$

nce we have that there exists an arbitrage for $(\bar{S}_t)$ iff there exists an
bitrage for $\tilde{S}_t$.
e usefulness of this observation is that if $S_t^0 > 0$ $\forall t$ then we can use $S^0$
$N$ on. In that case $\hat{S}_t = \frac{\bar{S}_t}{S_t^0} = \left(1, \frac{S_t^1}{S_t^0}, \dots, \frac{S_t^N}{S_t^0}\right)^T$

ns this puts us in domain of validity of FTAP, so that if there is no
bitrage then there exists $\mathbb{Q} \sim \mathbb{P}$ s.t. $\tilde{S}_t = \frac{\bar{S}_t}{S_t^0}$ is a $\mathbb{Q}$-martingale.

ecial case: $S_t^0 = (1+r)^t$ $(t \ge 0)$. This is a constant interest bank account.
en $\frac{\bar{S}_t}{(1+r)^t}$ is a $\mathbb{Q}$-martingale.

f. $S_t^2 = \frac{1}{S_t} \mathbb{E}_t \left[ S_t \, S_t \right]$

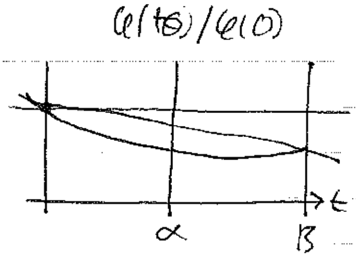
We saw the example of $S_t^0 = (1+r)^t$ and $S^1$ is a stock, $S^2$ is a European call option 14/10/11
aying $(S_T^1 - K)^+$ at time $T$.
, FTAP, $\tilde{S}_t^2 \equiv \frac{S_t^2}{S_t^0} \equiv \frac{S_t^2}{(1+r)^t}$ is a $\mathbb{Q}$-martingale, so

$$\tilde{S}_t^2 = \mathbb{E}^{\mathbb{Q}}\left[ \tilde{S}_T^2 | \mathcal{F}_t \right] = \frac{S_t^2}{(1+r)^t} = \mathbb{E}^{\mathbb{Q}}\left[ \frac{S_T^2}{(1+r)^T} \Big| \mathcal{F}_t \right]$$

Now notice that $\mathbb{Q} \ll \mathbb{P}$ on each $\mathcal{F}_t$, so there exists an $\mathcal{F}_t$-measurable density
$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \equiv M_t$, which is easily seen to be a $\mathbb{P}$-martingale because for $A \in \mathcal{F}_t$,

$$\int_A M_t \, d\mathbb{P} = \int_A \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} d\mathbb{P} = \mathbb{Q}(A) = \int_A \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_{t+1}} d\mathbb{P} = \int_A M_{t+1} \, d\mathbb{P} \quad \text{since } A \in \mathcal{F}_{t+1}$$

e use the FIP of the closed subsets $(F_\alpha)$ of the compact $S^{N-1}$; if $\bigcap_{\alpha>0} F_\alpha = \emptyset$, then $F_a = \emptyset$ for $a$.




$F_a = \emptyset$, then $\frac{\varphi(a\theta)}{\varphi(0)} > 1$ for all $\theta \in S^{N-1}$.

$\inf_{|\theta| \ge a} \frac{\varphi(\theta)}{\varphi(0)} > 1$ by convexity of $\varphi$.

the inf. of $\varphi(\theta)$ is attained. $\#$

must be that $\bigcap_\alpha F_\alpha \ne \emptyset$. This means there is some $v \in S^{N-1}$ such that for all $t \ge 0$ $\varphi(tv) \le \varphi(0)$

$\mathbb{E}\exp\left(-tv \cdot (S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2\right) \le \mathbb{E}\left[\exp\left(-\frac{1}{2}|S_1 - S_0|^2\right)\right]$

$\mathbb{P}[v \cdot (S_1 - S_0) < 0] = 0$, otherwise, as $t \to \infty$, the LHS gets unboundedly large.

$\mathbb{P}[v \cdot (S_1 - S_0) = 0] < 1$, so $\mathbb{P}[v \cdot (S_1 - S_0) \ge 0] = 1$, $\mathbb{P}[v \cdot (S_1 - S_0) > 0] > 0$.

is an arbitrage! □

We can likewise easily show that for $t \le u$, $Z \in \mathcal{F}_u$

$$\mathbb{E}^{\mathbb{Q}}[Z | \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}}[Z M_u | \mathcal{F}_t]}{M_t}$$

Thus we see that $\frac{S_t^2}{(1+r)^t} = \frac{1}{M_t} \mathbb{E}^{\mathbb{P}}\left(\frac{S_T^2}{(1+r)^T} M_T \,\Big|\, \mathcal{F}_t\right)$

So if we set $\boxed{\zeta_t \equiv M_t (1+r)^{-t}}$ we set $\boxed{S_t^2 = \frac{1}{\zeta_t} \mathbb{E}\left[\zeta_T S_T^2 | \mathcal{F}_t\right]}$ box again!

Proof of the FTAP Let's just do the case $T=1$; we'll suppose that $S_t^0 = 1$ for all $t$; and we'll suppose that the problem is non-degenerate in the sense that

$\mathbb{P}[\theta \cdot (S_1 - S_0) = 0] < 1$ for all $\theta \in \mathbb{R}^N$, $\theta \ne 0$.

This is an innocent assumption; if it fails, some assets are linearly dependent on each other, so we can discard the redundant assets.

Sketch of the idea $\sup_{\theta} \mathbb{E}\left[-\exp(-\theta \cdot (S_1 - S_0))\right]$ $\qquad U(x) = -e^{-x}$

If we can find $\theta^*$ which attains the sup. then we can differentiate w.r.t. $\theta$.

$$0 = \mathbb{E}\left[(S_1 - S_0) \underbrace{\exp(-\theta^* \cdot (S_1 - S_0))}_{\propto \text{ density of EMM}}\right]$$

To begin with, we clear the ground - we shall define

$$\theta \mapsto \varphi(\theta) = \mathbb{E}\left[\exp\left\{-\theta \cdot (S_1 - S_0) - \frac{1}{2}(S_1 - S_0)^2 - \lambda \xi - \frac{1}{2}\xi^2\right\}\right]$$

where $\xi$ could be any random variable, $\lambda$ any real.

We need this for $2^{nd}$ FTAP; but for now, we can think that $\xi = 0$

Proof (continued) The function $\varphi : \mathbb{R}^N \to \mathbb{R}$ is bounded below, so we consider the problem $\inf_{\theta \in \mathbb{R}^N} \varphi(\theta)$.

Two possibilities arise:

i) The inf is attained at $\theta^*$; in that case the (differentiable) function $\varphi$ has a vanishing derivative at $\theta^*$.

$$0 = D\varphi(\theta^*) = \mathbb{E}\left[(S_0 - S_1) e^{-\theta^*(S_1 - S_0) - \frac{1}{2}|S_0 - S_1|^2}\right]$$

So if we define $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\exp(-\theta^* \cdot (S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2)}{\mathbb{E}(\exp(-\theta^* \cdot (S_1 - S_0) - \frac{1}{2}|S_1 - S_0|^2))}$

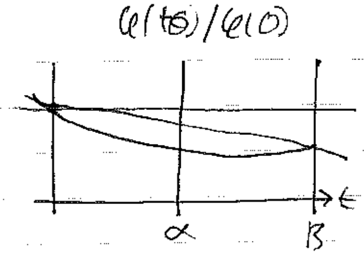
Then $\mathbb{E}^{\mathbb{Q}}(S_1 - S_0) = 0$, that is, $S_0 = \mathbb{E}^{\mathbb{Q}}(S_1)$, that is, $S_1$ is a $\mathbb{Q}$-martingale.

ii) The inf is not attained; so there exist $\theta_n$, $|\theta_n| \to \infty$ such that $\varphi(\theta_n) \downarrow \inf \varphi(\theta)$.

Let's consider for $\alpha > 0$ the set $F_\alpha$ defined by $F_\alpha = \left\{\theta : |\theta| = 1, \frac{\varphi(\alpha\theta)}{\varphi(0)} \le 1\right\}$

This is a closed subset of $S^{N-1}$.

We also see, from the convexity of $\varphi$, that for $0 < \alpha < \beta$ we must have $F_\beta \subseteq F_\alpha$.




we use the FIP of the closed subsets $(F_\alpha)$ of the compact $S^{N-1}$; if $\bigcap_{\alpha>0} F_\alpha = \emptyset$, then $F_a = \emptyset$ for $a$.

if $F_a = \emptyset$, then $\dfrac{\varphi(a\theta)}{\varphi(0)} > 1$ for all $\theta \in S^{N-1}$.

$\displaystyle\inf_{|\theta| \geq a} \frac{\varphi(\theta)}{\varphi(0)} > 1$ by convexity of $\varphi$.

the inf. of $\varphi(\theta)$ is attained. ※

it must be that $\bigcap_\alpha F_\alpha \neq \emptyset$. This means there is some $v \in S^{N-1}$ such that for all $t \geq 0$ $\varphi(tv) \leq \varphi(0)$

is, $\mathbb{E}\exp\left(-tv\cdot(S_1-S_0) - \frac{1}{2}|S_1-S_0|^2\right) \leq \mathbb{E}\left[\exp\left(-\frac{1}{2}|S_1-S_0|^2\right)\right]$

fore, $\mathbb{P}[v\cdot(S_1-S_0)<0] = 0$, otherwise, as $t \to \infty$, the LHS gets unboundedly large.

$\mathbb{P}[v\cdot(S_1-S_0) = 0] < 1$, so $\mathbb{P}[v\cdot(S_1-S_0) \geq 0] = 1$, $\mathbb{P}[v\cdot(S_1-S_0) > 0] > 0$.

$v$ is an arbitrage! $\square$