AFM

Math Finance approach to pricing.

In discrete time we have $N+1$ traded assets, the price at time t of asset i is denoted S_t^i, $i = 0, 1, \dots, N$, $t = 0, 1, \dots, T$.
We write

$$S_t = (S_t^1, S_t^2, \dots, S_t^N)^T, \quad \bar{S}_t (S_t^0, \dots, S_t^N)^T$$

Agents may buy or sell any number of units of these assets at those prices.
At time $t-1$ an agent chooses $\bar{\theta}_t = (\theta_t^0, \theta_t^1, \dots, \theta_t^N)$ of holdings of the $N+1$ assets into period t.
Thus θ_t is $\mathcal{F}_{t-1}$ measurable, so the process θ_t is previsible. The value of the portfolio at time t is

$$w_t = \bar{\theta}_t \cdot \bar{S}_t = \sum_{i=0}^{N} \theta_t^i S_t^i$$

We shall only look at <u>self-financing</u> portfolios processes $\bar{\theta}$, which satisfy

$$\bar{\theta}_t \cdot \bar{S}_t = \bar{\theta}_{t+1} \cdot \bar{S}_t$$ – "no cost" in changing portfolio.

If this happens, we see that

$$w_{t+1} - w_t = \bar{\theta}_{t+1} \cdot \bar{S}_{t+1} - \bar{\theta}_t \cdot \bar{S}_t$$

$$= \bar{\theta}_{t+1} \cdot (\bar{S}_{t+1} - \bar{S}_t) \equiv \bar{\theta}_{t+1} \cdot \Delta \bar{S}_{t+1}$$

So we find that

$$\boxed{w_t - w_0 = \sum_{j=1}^{t} \bar{\theta}_j \cdot \Delta \bar{S}_j}$$

the change in wealth is gains from trade.

The self-financing condition is a bit of a nuisance unless we had one of the assets (asset 0, say) which is always equal to 1. Because if that happens,

$$W_t - W_0 = \sum_{j=1}^{t} \theta_j \cdot \Delta S_j$$

because $(\Delta \bar{S})^0 \equiv 0$. This would mean that we could pick any previsible process θ and make $\bar{\theta}$ to be self-financing by setting θ^0_t to satisfy

$$\theta^0_t S^0_t + \theta_t \cdot S_t = \theta^0_{t+1} S^0_t + \theta_{t+1} \cdot S_t$$

$$\text{i.e.} \quad \boxed{\theta^0_{t+1} - \theta^0_t = -(\theta_{t+1} - \theta_t) \cdot S_t}$$

<u>Definition</u>: A portfolio $(\bar{\theta}_t)_{0 \le t \le T}$ is an arbitrage if

$$W_0 = \bar{\theta}_0 \cdot \bar{S}_0 \le 0, \quad 0 \le W_T \equiv \bar{\theta}_T \cdot \bar{S}_T \quad \text{and}$$

$$\mathbb{P}(W_T > 0) > 0$$

The basic axiom of arbitrage pricing theory is that the prices of assets are such that there is no arbitrage. Clearly, a system of equilibrium prices we would not get an arbitrage, but the concepts are <u>not</u> equivalent.

The big result is the following:

Theorem (Fundamental Theorem of Asset Pricing)

(a) $\exists$ a probability measure $Q \sim P$ such that the process $(\bar{S}_t)$ is a Q-martingale

Assume $S_0^t \equiv 1$ (b) There is no arbitrage.

Here $Q \sim P$ means $(Q(A) = 0) \Leftrightarrow (P(A) = 0)$ for any event A.

Same as: there exists a strictly positive $\mathbb{P}$-integrable Z st. $Q(A) = \int_A Z \, d\mathbb{P}$ (Radon-Nikodym thm)

So eg. if S_t^1 is a stock and S_t^2 an option on it then $S_t^2 = \mathbb{E}^Q(S_T^2 \mid \mathcal{F}_t)$

$$= \mathbb{E}^Q((S_T^1 - K)^+ \mid \mathcal{F}_t)$$

Remarks: Let us suppose there is some positive (strictly) adapted process $(N_t)_{0 \le t \le T}$ (meaning N_t is $\mathcal{F}_t$-meas.) and we define

$$\tilde{S}_t^i = \frac{S_t^i}{N_t}$$

$\tilde{W}_t = \frac{W_t}{N_t}$. Then $\tilde{W}_{t+1} - \tilde{W}_t = \frac{W_{t+1}}{N_{t+1}} - \frac{W_t}{N_t} =$

$$= \frac{\bar{\theta}_{t+1} \cdot \bar{S}_{t+1}}{N_{t+1}} - \frac{\bar{\theta}_t \cdot \bar{S}_t}{N_t}$$

$$= \bar{\theta}_{t+1} \cdot \left(\frac{\bar{S}_{t+1}}{N_{t+1}} - \frac{\bar{S}_t}{N_t} \right)$$

$$= \bar{\theta}_{t+1} \cdot \left(\tilde{S}_{t+1} - \tilde{S}_t \right)$$

Hence we have that $\exists$ an arbitrage for $(\bar{S}_t)$ iff $\exists$ an arbitrage for $\tilde{S}_t$.
The usefulness of this observation is that if $S_t^0 > 0 \ \forall t$ then we can use S^0 as N and in that case $\tilde{S}_t = \frac{\bar{S}_t}{S_t^0} = \left(1, \frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^N}{S_t^0}\right)^T$

Doing this puts us in domain of validity of FTAP, so that if there is no arbitrage then there exists $\mathbb{Q} \sim \mathbb{P}$ st.

$$\tilde{S}_t = \frac{\bar{S}_t}{S_t^0} \quad \text{is a } \mathbb{Q}\text{-martingale.}$$

Special case: $S_t^0 = (1+r)^t$ $(t \geq 0)$, constant interest bank account.

Then $\frac{\bar{S}_t}{(1+r)^t}$ is a $\mathbb{Q}$-martingale.

Cf. $S_t^2 = \frac{1}{\zeta_t} \mathbb{E}_t\left[\zeta_T \, S_T^2\right]$