$$\int_0^t \frac{1}{2}\Sigma_{uT}^2\, du = \int_0^t \left(\int_0^T \alpha_{us}\, ds\right) du$$

$$\Rightarrow \quad \sigma_{uT} = \frac{\partial \Sigma_{uT}}{\partial T},$$

$$\frac{1}{2}\Sigma^2_{uT} = \int_0^T \alpha_{us}\, ds$$

$$\Sigma_{ut}\frac{\partial \Sigma_{uT}}{\partial T} = \alpha_{uT}$$

$$\Rightarrow \quad \alpha_{uT} = \sigma_{uT}\,\Sigma_{uT} = \sigma_{uT}\int_0^T \sigma_{us}\, ds$$

Remarks. (i) So we see that once the vol. structure σ_{st} for the forward rates is chosen, no arbitrage tells us the drift α.

(ii) This is the main (only?) result in HJM's paper.

(iii) It's relatively easy to build Gaussian examples here, just try taking σ to be deterministic ... but rates can go negative ...

(iv) If you want a model where $r_t \geq 0$, it's quite hard to do in this framework.

Interest-rate derivatives.

1) <u>Caps</u> The idea here is that each time $T_1 < T_2 < \cdots < T_n$ you have to make interest payments. The payment at time T_{j+1} is determined by the LIBOR rates at time T_j:

$$B(T_j, T_{j+1}) = \frac{1}{1+(T_{j+1}-T_j)L(T_j, T_{j+1})}$$

$$\Rightarrow \quad (T_{j+1}-T_j)L(T_j, T_{j+1}) = \frac{1}{B(T_j, T_{j+1})} - 1 .$$

A <u>caplet</u> will give you

$$((T_{j+1}-T_j)L(T_j, T_{j+1}) - k)^+$$

at time T_{j+1}.
The value of the caplet at Time T_j is

$$\mathbb{E}_{T_j}\left(e^{-\int_{T_j}^{T_{j+1}} r_s\, ds} \left((T_{j+1}-T_j)L(T_j, T_{j+1}) - k \right)^+ \right)$$

$$= B(T_j, T_{j+1}) \left(\frac{1}{B(T_j, T_{j+1})} - 1 - k \right)^+$$

$$= (1-(1+k)B(T_j, T_{j+1}))^+$$

So the caplet is worth the same as a put option on the ZCB paying at T_{j+1}.
Notice that actually evaluating the time-0 price of this,

$\mathbb{E}[\exp(-\int_0^{T_j} r_s\, ds)(1-(1+k)B(T_j, T_{j+1}))^+]$
— these things are <u>not</u> easy to calculate....

If you can't do it properly, you fake it, and here we often find Black-Scholes applied... this is ~~not~~ quite unsound but widespread.

$$E[\cdots] = \mathbb{E}^{T_j}[(1-(1+k)B(T_j, T_{j+1}))^+] \, B(0, T_j).$$

2) <u>Swaps</u> The idea of a swap is to exchange floating interest payments for fixed. The fixed log of this swap pays interest at some fixed rate k at times $T_1 < \cdots < T_n$. The values of these payments will be

$$\sum_{j=1}^{n} k(T_j - T_{j-1}) B(0, T_j)$$

As for the floating payments, you pay

$$L(T_{j-1}, T_j)(T_j - T_{j-1}) \text{ at time } T_j.$$

The time-0 value of this is
$$1 - B(0, T_n).$$

This allows us to calculate the <u>par swap rate</u> k, which is the rate which would equalize the values of the two logs:

$$k = \frac{1 - B(0, T_n)}{\sum_{j=1}^{n}(T_j - T_{j-1}) B(0, T_j)}$$

3) <u>Swaptions</u> : All this is an option to enter into a swap; you have to specify the exercise date T^*, the strike k, and the <u>tenon structure</u>.

$$T^* \equiv T_0 < T_1 < \cdots < T_n,$$

What you would get for the time-0 price of this would be

$$\mathbb{E}\left[\left(1 - B(T^*, T_n) - k \sum_{j=1}^{n} (T_j - T_{j-1}) B(T^*, T_j)\right)^+ \cdot \exp\left(-\int_0^{T^*} r_s \, ds\right)\right]$$

— quite a mess ...

<u>Consider this derivative</u> :

Pay quarterly a fixed in USD, receive floating GBP until CHF / EUR > 1.6 (Suppose we're accounting in USD).

How are we going to do this? Go back to box, and model <u>only</u> what we need, namely, the state-price density ζ:

$$Y_t = \frac{1}{\zeta_t} E_t[Y \zeta_t]$$

We know that

$$\zeta_t = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} e^{-\int_0^t r_s \, ds}$$

If $r \geq 0$, then ζ is a supermartingale, and positive. How could we build a reasonably flexible class of positive supermartingales? Pick some Markov process X_t, and use this as the basis of the modelling effort. and then ~~by~~ try to express

$$\zeta_t = f(X_t) \exp\left(-\int_0^t \alpha(X_s)\, ds\right)$$

So ~~we~~ when can we say that this thing is a supermartingale? If X were a BM, we can do Itô:

$$d\zeta = e^{-\int_0^t \alpha(X_s)\, ds}\left[-\alpha(X_t) f(X_t)\, dt + f'(X_t)\, dX_t + \tfrac{1}{2} f''(X_t)\, dt\right]$$

$$= d\,\text{mart} + e^{-\int_0^t \alpha(X_s)\, ds}\left[\tfrac{1}{2} f''(X_t) - \alpha(X_t) f(X_t)\right] dt$$

Provided $\alpha f \geq \frac{1}{2} f''$ and $f \geq 0$, we've got a supermartingale.

⌈ Analogously, if X is a finite-state Markov Chain with jump intensity matrix Q, we get

$$d\zeta_t = d\,\text{mart} + \left[(Qf)(X_t) - \alpha(X_t) f(X_t)\right] dt$$

(think of f as column vector, $f = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$)

– the reason this is worth considering is that you only ever need to do finite sums to calculate option prices ... ∃ ⌋

So what we could do is propose a model via

(i) the Markov process X
(ii) the non-negative function α
(iii) $g = \alpha f - \frac{1}{2} f'' \geq 0$.

— Choose g, and solve for f.

Given these things, we have a pricing framework. We also know that

$$d\zeta_t = d\,mart - r_t \zeta_t\, dt$$
$$= \exp\left(-\int_0^t \alpha(X_s)ds\right)\left(d\,mart - (\alpha f - \tfrac{1}{2} f'')(X_t)\, dt\right)$$

$$\Rightarrow \quad \boxed{r_t = \frac{\alpha f - \frac{1}{2} f''}{f}(X_t)}$$

How about multiple currencies?

Suppose S_t^j is a traded asset in currency j, let Y_t^{ij} be the exchange rate at time t from currency j to currency i:

1 unit of currency j = Y_t^{ij} units of currency i.

In country i, we have a state-price density ζ_t^i such that

$(\zeta_t^i \cdot S_t^i)$ is a martingale.

So asset S^i express in currency j is $Y_t^{ji} S_t^i$, and this is a currency j traded asset.

So $\zeta_t^j Y_t^{ji} S_t^i \equiv \frac{\zeta_t^j Y_t^{ji}}{\zeta_t^i} (\zeta_t^i S_t^i)$ is a martingale.

Thus what we find is that

$\zeta_t^j Y_t^{ji} / \zeta_t^i \equiv N_t$ is a martingale orthogonal

to all martÞ of the form $\zeta^i S^i$, and if the market is <u>complete</u>, then N must be <u>constant</u>.

$$Y_t^{ji} = \frac{\zeta_t^i}{\zeta_t^j} \cdot Y_0^{ji}$$

— in such a framework, adding another currency does not (or does not need to) change the underlying Markov process X; we just have to pick some new α_j, g_j, and we ~~hence can Si~~ have <u>consistently and simultaneously</u> modeled interest rates in many countries and the exchange rates between the currencies.

Re: choosing g and α:

$$f(x) = \mathbb{E}^{x} \left[\int_0^\infty e^{-\int_0^t \alpha(X_s)\,ds} g(X_t)\,dt \,\middle|\, X_0 = x \right]$$

Credit

A simple credit derivative: CDS = credit default swap.

You hold a corporate bond which promises to pay δ_i at time T_i, $i=1,\dots,n$

You want protection against default and a CDS gives this.

If τ is the time of default of the corporation, the CDS pays you

$$\sum_{\tau_j \geq \tau} \delta_j \, B(\tau, \tau_j)$$

at τ.

How would we price such a thing? We need an interest rate model for the bonds, and we also need a model for the default time τ, but we can't assume independence of τ, r_t.

Let's suppose we've got some story for this (you can fit some tale from market data).

How about hedging? The only hedging instrument is a short stock... and this is problematic... Delta-hedging won't work? How about more complicated credit derivatives where payoffs depend on default of many entitles (CDOs).