Vasicek: (dB_t)

$$dr_t = \sigma dW_t + \lambda(r_\infty - r_t)\,dt$$

$$r_t = r_\infty + e^{-\lambda t}(r_0 - r_\infty) + \int_0^t e^{\lambda(s-t)}\sigma\, dB_s$$

We find that the bond prices to be the form

$$B(t,T) = \exp(-r_t\, a(T-t) - b(T-t))$$

This tells us that the possible shapes of the yield curve are very restricted, once we know $\lambda, \sigma, r_\infty$, all that can happen is that r changes. So it would be difficult to fit the shape of observed market data.

But if we were to consider a different SDE for the short rate; set $\tilde{r}_t = r_t + h(t)$ where is C^1, available to be chosen.

We see then that

$$d\tilde{r}_t = \sigma dB_t + \lambda(r_\infty - (\tilde{r}_t - h(t)))\,dt$$

– like Vasicek, but the mean reversion level r_∞ has become a function of time, $r_\infty + h(t)$.

So all that happen is that $-\log B(t,T)$ gets changed from

Then $\tilde{r}_t = r_t + h(t)$

$$\Rightarrow \int_t^T \tilde{r}_s\, ds = \left(\int_t^T r_s\, ds\right) + \left(\int_t^T h(s)\,ds\right) \qquad \mathbb{E}_t\, e^{-\int_t^T \tilde{r}_s ds} = \left(\mathbb{E}_t\, e^{-\int_t^T r_s ds}\right) \cdot e^{-\int_t^T h(s)ds}$$

So $-\log B(t,T)$ is changed by the addition of $\int_t^T h(s)\,ds$ and can be adjusted to match the market data at day t.

Of course, if you match on day t through one ~~parameter~~ particular choice of h, there's no guarantee that the same h will work at time $t+\varepsilon$. (Hull – White)

Cox – Ingersoll – Ross model

This is a one-factor spot rate model, where r evolves like

$$dr_t = \sigma\sqrt{r_t}\, dB_t + (a - b r_t)\,dt$$

The aim is to calculate a closed-form expression for the bond price,

$$B(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s\, ds} \,\middle|\, \mathcal{F}_t\right]$$

what we have is that

$$\mathbb{E}[\exp(-\int_0^t r_s ds)] = \varphi(t, r_0)$$

We know that if we fix T

$$M_t = \mathbb{E}[\exp(-\int_0^T r_s ds) | \mathcal{F}_t]$$
$$= \exp(-\int_0^t r_s ds)\, \varphi(T-t, r_t)$$

is a martingale. We do Itô

$$dM_t = e^{-\int_0^t r_s ds}\{-r_t dt\, \varphi - \dot{\varphi} dt + \varphi_r dr + \tfrac{1}{2}\varphi_{rr} dr dr\}$$

The drift must be zero, so we deduce

$$0 = -r\varphi - \dot{\varphi} + \varphi_r(a-br) + \tfrac{1}{2}\varphi_{rr}\sigma^2 r$$

We have to solve this PDE with boundary condition $\varphi(0,r)=1$

It turns out that the solution has the form

$$\varphi(t,r) = \exp(-rA(t) - C(t)\}$$

where we must have $A(0) = \cancel{B(0)} = 0 = C(0)$. The PDE now looks like $\quad 0 = -r + (r\dot{A} + \dot{C}) - (a-br)A + \frac{1}{2}\sigma^2 r A^2$

which amounts to

$$\begin{cases} 0 = -1 + \dot{A} + bA + \frac{1}{2}\sigma^2 A^2 \\ 0 = \dot{C} - aA \end{cases}$$

$$\frac{dA}{\frac{1}{2}\sigma^2 A^2 + bA - 1} = -dt$$

$$= \frac{dA}{\frac{1}{2}\sigma^2 (A-\theta_1)(A-\theta_2)} \qquad \text{where } \theta_1, \theta_2 \text{ are roots of } \tfrac{1}{2}\sigma^2 x^2 + bx - 1 = 0$$

$$= \frac{2dA}{\sigma^2}\left(\cancel{+}\frac{1}{A-\theta_1} - \frac{1}{A-\theta_2}\right)\frac{1}{\theta_1-\theta_2}$$

$$\Rightarrow \frac{2}{\sigma^2(\theta_1-\theta_2)}\log\left(\frac{A-\theta_1}{A-\theta_2}\right) = -t + k$$

so we get $\quad \dfrac{A-\theta_1}{A-\theta_2} = K\exp(-\gamma t) \quad$ where $\gamma = \dfrac{\sigma^2(\theta_1-\theta_2)}{2} > 0$ wlog

We know $A(0)=0$, so $K = \theta_1/\theta_2$, and

$$\frac{A-\theta_1}{A-\theta_2} = \frac{\theta_1}{\theta_2}\cdot e^{-\gamma t} \Rightarrow A(t) = \frac{\theta_1 - \theta_1 e^{-\gamma t}}{1 - \theta_1/\theta_2 e^{-\gamma t}} = \frac{\theta_2\theta_1(1-e^{-\gamma t})}{\theta_2 - \theta_1 e^{-\gamma t}}$$

Then we get $C(t) = a\int_0^t A(s)\,ds$

which can be done in closed form also.

This is ~~at~~ really all that's good about CIR:

- what about closed-form expressions for other interest-rate derivatives?
- the dependence of vol on r is just not right

$$dr = \sigma r\, dB + \text{—}$$

is much closer ...

- it's a 1-factor model ... so all yields of all maturities are instantaneously perfectly correlated.

Black-Karasinski:

$$dr_t = \sigma r_t\, dB_t + \mu r_t\, dt$$

So the riskless rate evolves like a log BM. The bond price would be

$$E[\exp(-\int_0^T r_s\, ds)]$$

which is not known in closed form ...

(it's a 1-factor model, all the criticisms apply)

Why not one-factor models?

Suppose we go and gather some data on yields on day t, we record

$$Y_t^j \equiv -\frac{1}{T_j}\log B(t, t+T_j)$$

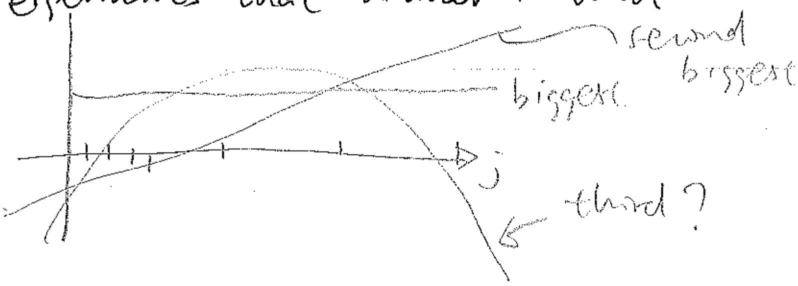
$j = 1, 2, \cdots, J$. Now we consider Y_t as a J-vector, and we can do some basic statistics, like calculate the sample means

$$\bar{Y}_t = \frac{1}{t}\sum_{i=1}^{t} Y_i$$

and sample variance

$$V_t = \frac{1}{t}\sum_{i=1}^{t}(Y_i - \bar{Y}_t)(Y_i - \bar{Y}_t)^T$$

and then look at the eigenstructure of V_t. Typically, we find that there are 2-3 eigenvalues that 'matter', with corresponding eigenvectors



A 1-factor model could not generate such dynamics ...