Leture 2

$$\Pi_{st}(Y) = \frac{1}{\zeta_s}\mathbb{E}(\zeta_t Y)$$

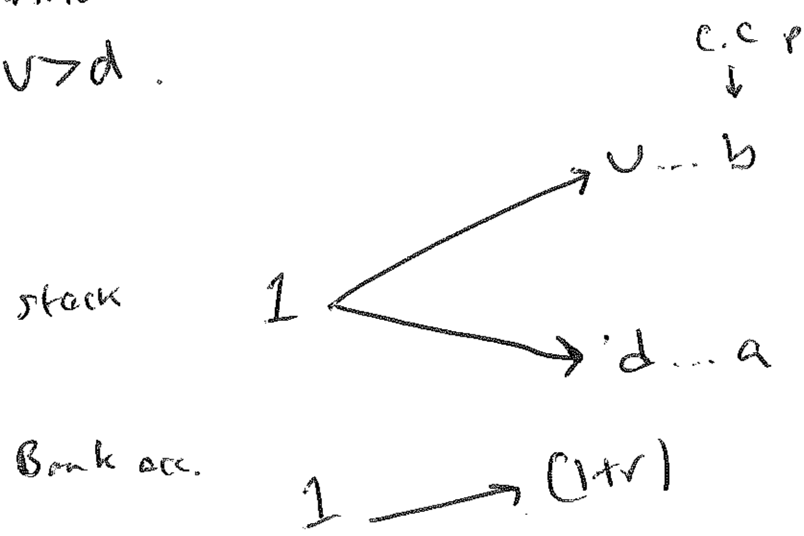
Not in general ok to say $\Pi_{0t} = \mathbb{E}(Y)$

Because time-0 price of 1 for sure at time t is e^{-rt} if there's a bank account.

But still not enough to have

$$\Pi_{0t} = \mathbb{E}(e^{-rt} Y)$$

Why? Consider a situation where we have two assets, a stock $S_0 = 1$ and a bank account $B_0 = 1$, and at time 1 the bank account is $1+r$. And the stock is worth either u or d, where $u > d$.



consider a contingent Claim (r.v.) which pays $\begin{cases} b & \text{if } S_1 = u \\ a & \text{if } S_1 = d \end{cases}$

What is the time-0 price of this contingent claim? Suppose at time-0 we hold x units of stock and y units of the bank account.

At time 1, we have
$$\begin{cases} \underset{\#\text{ stock} = x}{x}u + \underset{\text{units bank}}{y}(1+r) = b & \text{if } S_1 = u \\ xd + y(1+r) = a & \text{if } S_1 = d \end{cases}$$

So we can solve for X, Y:

$$x = \frac{b-a}{u-d}, \quad y = \frac{1}{1+r}\left(b - \frac{(b-a)u}{u-d}\right)$$

$$= \frac{1}{1+r}\left(\frac{au - bd}{u-d}\right)$$

So the <u>only</u> price possible time-0 price for this contingent claim ~~B~~ is

$$x + y = \frac{b-a}{u-d} + \frac{1}{1+r}\left(\frac{au-bd}{u-d}\right)$$

$$= \frac{(1+r)(b-a) + au - bd}{(u-d)(1+r)}$$

$$= \frac{a(u-1-r) + b(1+r-d)}{(u-d)(1+r)}$$

$$= \frac{1}{1+r}\left(pb + (1-p)a\right)$$

$$\boxed{\text{where } p = \frac{1+r-d}{u-d}}$$

The interpretation ~~for this~~ is that this is

$$\frac{1}{1+r}\mathbb{E}(Y) \qquad \text{where } \tilde{\mathbb{P}}(S_1 = u) = p$$

So we see that $\boxed{\Pi_{st}(Y) = \frac{1}{\zeta_s}\mathbb{E}(\zeta_t Y)}$

is still valid in this simple story and takes the form

$$\Pi_{0t}(Y) = \frac{1}{1+r}\tilde{\mathbb{E}}(Y)$$

So if p_0 is the true probability that $S_1 = u$

then $\zeta_1 = \frac{1}{1+r}\left\{\frac{p}{p_0} I_{\{S_1=u\}} + \frac{1-p}{1-p_0} I_{\{S_1=d\}}\right\}$

objective probability (real world probability)

every time we wish to calculate a price we should calculate an expectation...

Economists approach the Lucas tree

Story is this. There is a single tree which in period t (discrete time) produces a quantity δ_t of fruit, which can only be consumed in period t. There are agents in this economy, who have preferences over consumption streams $(c_t)_{t\geq 0}$ given by $U_j(c) = \mathbb{E}\left[\sum_{t\geq 0} u_j(t, c_t^j)\right]$

agent j

where each $u_j(t,\cdot)$ is strictly increasing, strictly concave.

Suppose that in period t, the ex dividend price of the tree is S_t, and in period t an agent chooses to hold θ_{t+1} units of the tree, and chooses to consume c_t, subject to the budget constraint $\theta_t(\delta_t + S_t) = c_t + \theta_{t+1} S_t$

(in units of period-t fruit)

So this is the only constraint. Write the agents problem in Lagrangian form.

$$\max \mathbb{E}\Big[\sum_{t\geq 0} U(t, c_t) + \sum_t \lambda_t \big(\theta_t(\delta_t + S_t) - c_t - \theta_{t+1} S_t\big)\Big]$$

$$= \max \mathbb{E}\Big[\sum_{t\geq 0} \big(U(t, c_t) - \lambda_t c_t\big) + \sum_{t\geq 1} \theta_t \big(\lambda_t (\delta_t + S_t) - \lambda_{t-1} S_{t-1}\big) + \lambda_0 \theta_0 (\delta_0 + S_0)\Big]$$

Now we optimise over c, θ:

$$U'(t, c_t) = \lambda_t$$

So that's easy: the maximisation over c gives

$$U(t, c_t) - \lambda_t c_t = \tilde{U}(t, \lambda_t)$$

where $\tilde{U}(y) = \sup(U(x) - yx)$

For maximisation over θ_t, we have

$$\mathbb{E}\left[\theta_t\left\{\lambda_t(S_t+\delta_t)-\lambda_{t-1}S_{t-1}\right\}\right]$$

Hence,

$$= \mathbb{E}\left[\theta_t\,\mathbb{E}_{t-1}\left(\lambda_t(S_t+\delta_t)-\lambda_{t-1}S_{t-1}\right)\right]$$

Since θ_t is $\mathcal{F}_{t-1}$ meas.

In order that the sup over θ_{t-1} be finite, we need the dual feasibility condition

$$0=\mathbb{E}_{t-1}\left(\lambda_t(S_t+\delta_t)-\lambda_{t-1}S_{t-1}\right) \quad \forall\, t\geq 1$$

But λ_{t-1}, S_{t-1} are $\mathcal{F}_{t-1}$-meas. so this says

$$S_{t-1}=\frac{1}{\lambda_{t-1}}\mathbb{E}_{t-1}\left(\lambda_t\delta_t+\lambda_t S_t\right)$$; or if you prefer

$$\lambda_{t-1}S_{t-1}=\mathbb{E}_{t-1}\left(\lambda_t\delta_t+\lambda_t S_t\right)$$

$$=\mathbb{E}_{t-1}\left(\lambda_t\delta_t+\lambda_{t+1}\delta_{t+1}+\lambda_{t+1}S_{t+1}\right)$$

$$=\mathbb{E}_{t-1}\left(\sum_{j\geq t}\lambda_j\delta_j\right)$$

This agent facing these prices would choose

$$\boxed{c_t=(U')^{-1}(t,\lambda_t)}$$

and then

$$S_{t-1}=\frac{1}{\lambda_{t-1}}\mathbb{E}_{t-1}\left(\sum_{j\geq t}\lambda_j\delta_j\right)$$

where do the λ_j come from? market clearing!

Faced with prices S_t, agent j with multiplier process λ^j will choose

$$c_t^j = (U_j')^{-1}(t, \lambda_t^{\xi j})$$

For simple situations, notably representative agent economies (and central planner economies) where there is just one agent.

$$\text{The total consumed} = \sum_j c_t^j = \underset{\text{consumption agent } 1}{c_t^1} = \delta_t$$

Given this, we know $c_t^1 = \delta_t = (U_1')(t, \lambda_t)$

So we know λ_t!!

Then we know the equilibrium price of the tree;

$$S_{t-1} = \frac{1}{\lambda_{t-1}} \mathbb{E}_{t-1}\left[\sum_{j \geq t} \lambda_j \delta_j\right]$$

Remarks:

- Notice that the price of the tree is given as

$$\underbrace{\sum_{j \geq t} \frac{1}{\lambda_{t-1}} \mathbb{E}_{t-1}(\lambda_j \delta_j)}$$

the time-$(t-1)$ price for δ_j to be delivered at time j. $(\lambda = 3)$ from prev.

Representative agent economies are unrealistic.

Multi-agent equilibria are usually hard to solve.

- but they do tell us how to deal with Contingent claims.