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This is a quite different story for many reasons. The most basic instrument is bond. This is often a government bond. The bond has a face value (which we ~~always~~ may as well take to be 1), and promises to repay the face value at maturity, and also pay coupons at regularly-spaced intervals. The basic component of this structure is a zero-coupon bond, which is an asset which will pay 1 at time T. The main difference between a zero coupon bond + a stock is that for a ZCB we know exactly what it is worth at T. (hence the name fixed income) So the time zero price of a ZCB is

$$B(0,T) = E^{Q}\left[\exp\left(-\int_0^T r_s\,ds\right) \cdot 1\right]$$

where the spot-rate process is some random process. The assumption in equity derivatives is that r can be treated as constant, but ~~[illegible]~~ that's for the reasons that

(i) typically expiries of equity derivatives are quite short

(ii) the order of fluctuations of ~~stock~~ bond prices tends to be ~~much~~ less more than in ~~equity~~ equity.

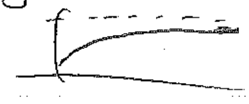
(iii) You can always price a 6-month using 6-month LIBOR.

Terminology: The yield-to-maturity $Y(t,T)$ for $t<T$ is

$$Y(t,T) = -\log(B(t,T)) \iff B(t,T) = e^{-Y(t,T)}.$$

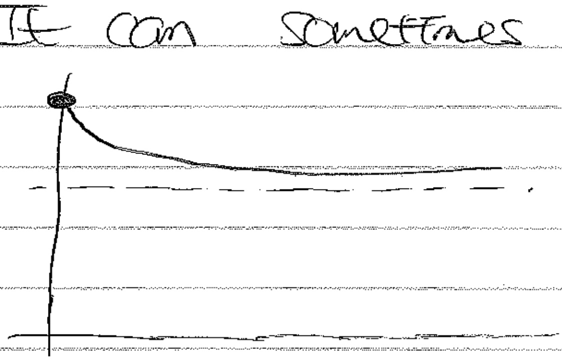
and the yield $y(t,T)$ is just $\frac{1}{T-t} Y(t,T)$.

The yield curve $u \mapsto y(t,t+u)$ $(u \geq 0)$ is an object of primary interest: Typically looks like.

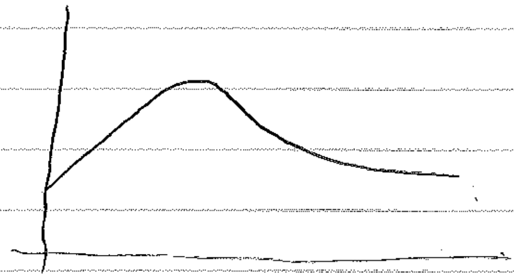


P.T.O

It can sometimes be <u>inverted</u>.



Or even humped



How would we go about modelling interest rates?

(i) Establish a model for the random process r.

(ii) Heath-Jarrow-Morton Approach:

$$-\log B(t,T) \equiv Y(t,T) = \int_t^T f_{ts}\,ds.$$

Where f_{ts} is the instantaneous forward rate at time t for maturity $s \geq t$.

(allow testing for sensitivity).

(iii) Then come the various market models:

$$\frac{1}{1+(T-t)L(t,T)} = B(t,T).$$

defines the <u>forward-rate</u> $L(t,T)$; the third approach proposes some dynamics for L.

<u>Some explicit example of interest rate models</u>

1) <u>The Vasicek model</u>

$$dr_t = \sigma\,dB_t + \lambda(r_\infty - r_t)\,dt.$$

where $\lambda > 0$, $\sigma > 0$, r_∞ are constants. We know this SDE has an explicit sol^n.

$$r_t = r_\infty + e^{-\lambda t}(r_0 - r_\infty) + \sigma\int_0^t e^{\lambda(s-t)}\,dB_s$$

So in this case r is a Gaussian Process, and

$$\int_0^T r_t\,dt = r_\infty T + (r_0 - r_\infty)\frac{1-e^{-\lambda T}}{\lambda} + \sigma\int_0^T\left(\int_0^t e^{\lambda(s-t)}\,dB_s\right)dt.$$

$$= r_\infty T + (r_0 - r_\infty)\frac{1-e^{-\lambda T}}{\lambda} + \sigma\int_0^T \frac{(1-e^{-\lambda(T-s)})}{\lambda} dB_s.$$

$$\sim N\left(r_\infty T + (r_0 - r_\infty)\underbrace{\frac{1-e^{-\lambda T}}{\lambda}}_{\mu_T}, \underbrace{\frac{\sigma^2}{\lambda^2}\int_0^T (1-e^{-\lambda u})^2 du}_{\nu_T}\right)$$

Hence in the Vasicek model, we see

$B(0,T) = \exp(-\mu_T + \frac{1}{2}\nu_T)$.

Notice that this is a f^n of r_0. What this means is that the yields to all maturities perfectly correlated

$d(y(t,T_j)) = \underbrace{e(t,T_j,r_t)\, dB_t}_{\text{same BM}} + \_\_\_\_\_\_$

All the stochastic integrals are driven by just 1 BM.

This conclusion is readily refuted by data:
(The same criticism damages any one-factor interest-rate model)

Another snag is that we [redacted] could get bond prices > 1.

... in 1977, it didn't really matter, but it's believed that negative interest rates are absurd! But they have happened...

— the use of this model now is limited to situations where there's a lot more going on & it allows some interest rate variability.

[illegible]