$\rightarrow dX_t = 2\sqrt{X_t}\, dB_t + n\,dt$

BESQ(n) If $\left.\begin{array}{l} X \text{ is } BESQ(\alpha) \\ X' \quad BESQ(\beta) \end{array}\right\}$ indep $\Rightarrow X+X'$ is BESQ $(\alpha+\beta)$

(Eg, if $\alpha=\beta=\frac{1}{2}$, X, X' indep, $X+X'$ hits 0 and spends no time ?!!

$$\left(\begin{aligned} d(X+X') &= 2\sqrt{X}\,dB + 2\sqrt{X'}\,dB' \\ &= 2(\sqrt{X}\,dB + \sqrt{X'}\,dB') = 2(\sqrt{X+X'}\,d\tilde{B}) \end{aligned} \right)$$

There are many reasons why BESQ processes and their relatives are important in the subject;

(i) The transition density of BESQ(δ) is known in closed form:

$$q_t(x,y) = \frac{1}{2t}\left(y/x\right)^{\nu/2} e^{-(x+y)/2t}\, I_\nu(\sqrt{xy}/t) \qquad [\nu \equiv \tfrac{\delta}{2}-1]$$

(ii) There are various related process, such as the CEV process;

$$\begin{aligned} dS &= \sigma S^\theta\, dB \qquad (0<\theta<1) \\ &= S(\sigma S^{\theta-1})\,dB \end{aligned}$$

We can relate this to BESQ processes by writing $Y = S^\beta$, and then

$$\begin{aligned} dY &= \beta S^{\beta-1}\,dS + \frac{\beta(\beta-1)}{2} S^{\beta-2}\,dS\,dS \\ &= \sigma\beta S^{\beta-1+\theta}\,dB + \frac{\beta(\beta-1)}{2}\sigma^2 S^{\beta-2+2\theta}\,dt \end{aligned}$$

If we take $\beta = 2(1-\theta)$, we get

$$d(S^\beta) \equiv dY = \sigma\beta S^{\beta/2}\,dB + \frac{\beta(\beta-1)}{2}\sigma^2\,dt$$

So $cY \equiv cS^\beta$ solves a BESQ SDE, so we know the transition density of S and so we can price European call options.

We have $dY = 2\sigma(1-\theta)\sqrt{Y}\,dB + (1-\theta)(1-2\theta)\sigma^2\,dt$

If $\sigma = 1/1-\theta$, we get

$$dY = 2\sqrt{Y}\,dB + \left(\frac{1-2\theta}{1-\theta}\right)dt$$

<1

This process will hit zero infinite time ... So perhaps not such a good model for a stock ... Have to make the asset stick at zero ...

(iii) The so-called <u>Cox-Ingersoll-Ross</u> process solves

$$dX_t = \sigma\sqrt{X_t}\,dB_t + (\alpha - \beta X_t)\,dt$$

How is this related to BESQ? Consider $Y_t = e^{\beta t} X_t$

$$d(Y_t) = e^{\beta t}(\beta X_t dt + dX_t)$$
$$= e^{\beta t}(\sigma\sqrt{X_t}\,dB_t + \alpha dt)$$
$$= \sigma e^{\beta t/2}\sqrt{Y_t}\,dB_t + \alpha e^{\beta t}\,dt$$

So $Y_t - Y_0 - \int_0^t \alpha e^{\beta s}\,ds = \int_0^t \sigma e^{\beta s/2}\sqrt{Y_s}\,dB_s$

and if we now write

$$A_t = \int_0^t e^{\beta s}\,ds, \quad \tau_t = \inf\{u : A_u > t\}$$

$$Y_{\tau_t} - Y_0 - \alpha t = \underbrace{\int_0^{\tau_t} \sigma e^{\beta s/2}\sqrt{Y_s}\,dB_s}_{\text{this a martingale with quadratic variation}}$$

$s = \tau_u$
$A_s = u$
$e^{\beta s}ds = du$

$$\int_0^{\tau_t} \sigma^2 e^{\beta s} Y_s\,ds = \int_0^t \sigma^2 Y_{\tau_u}\,du$$

So if $\tilde{Y}_u \equiv Y_{\tau_u}$, we have

$$d\tilde{Y}_u = \alpha\,du + \sigma\sqrt{\tilde{Y}_u}\,d\tilde{B}_u$$

That is, $\tilde{Y}$ solves a BESQ SDE.

<u>The Heston model</u>. The idea here is to make the vol (volatility) in the BS model a random process.

So we take

$$\begin{cases} dV_t = a\sqrt{V_t}\,dB_t + (\alpha - \beta V_t)\,dt \\ dS_t = \sqrt{V_t}\,S_t\,dB'_t \end{cases}$$

where B, B' are two BMs, with constant correlation.

$$dB\,dB' = \rho\,dt$$

[or: $dB' = \rho\,dB + \sqrt{1-\rho^2}\,d\tilde{B}$ for $\tilde{B}$ indept of B]

Given this model, we have an expression for the stock price at time t

$$S_t/S_0 = \exp\left(\int_0^t \sqrt{V_s}\,dB'_s - \tfrac{1}{2}\int_0^t V_s\,ds\right)$$
$$= \exp\left(\int_0^t \sqrt{V_s}(\rho\,dB_s + \rho'\,d\tilde{B}_s) - \tfrac{1}{2}\int_0^t V_s\,ds\right)$$

$\rho' \equiv \sqrt{1-\rho^2}$

$$= \exp\left[\frac{\rho}{a}\int_0^t \{dV_s - (\alpha - \beta V_s)ds\} + \rho'\int_0^t \sqrt{V_s}\,d\hat{B}_s - \frac{1}{2}\int_0^t V_s ds\right]$$

$$= \exp\left[\frac{\rho}{a}(V_t - V_0) - \int_0^t \left(\frac{1}{2}V_s + \frac{\rho}{a}(\alpha - \beta V_s)\right)ds + \rho'\underbrace{\int_0^t \sqrt{V_s}\,d\hat{B}_s}\right]$$

Conditional on V, this is $N(0, \int_0^t V_s ds)$

So if we can characterize the joint dist^n of $(V_t, \int_0^t V_s ds)$ then we know the law of S_t, and can calculate European option prices.

Exercise: $E[\exp\{-\lambda\int_s^T V_u du - \mu V_T\} | \mathcal{F}_s] = \exp(-A(s) - B(s)V_s)$

$\exp(-\lambda\int_0^t V_u du)\cdot\exp(-A(t) - B(t)V_t)$ is a martingale

$\rightarrow$ ODE for A, B (Riccatti or Ricatti ?)