which is a Gaussian r.v., we can calculate the conditional distr. of $B_t | Z$, we find

$$\mathbb{E}\, B_t | Z = m_t Z, \quad \mathrm{var}(B_t | Z) = v_t$$

m_t, v_t - some deterministic fns.

So we have

$$\mathbb{E}\Big[\int_0^T S_u \,\mu(du) \,\Big|\, Z \Big] = \int_0^T S_0 \exp\{ \sigma m_t Z + \tfrac{1}{2}\sigma^2 v_t + (r - \tfrac{1}{2}\sigma^2)t \} \,dt$$

So we can calculate a lower bound by numerical integration, two dimensional. It is amazingly close to true value (errors of order 1%)

This says $\int_0^T \exp(\sigma B_t + \nu t)\,dt$

$$\approx \int_0^T (1 + \sigma B_t + \nu t)\,dt$$

Thus the conditioning variable Z captures well the randomness of Z.

We can also get nice upper bounds:

$K = \int_0^T k_t \,\mu(dt)$. Then we have

$$\mathbb{E}\Big(\int_0^T (S_u - k_u)\,du \Big)^+ \leq$$

$$\leq \mathbb{E} \int_0^T (S_u - k_u)^+ \,du.$$ This we can minimise over k.

Exercise: Prove that for some α, k_u is the α-quantile of S_u.

You can also combine this trick with condition to Z (Giles Thompson). The upper & lower bounds are typically very good.

So we have good approximations here...
and for American put still probably best to do Crank-Nicholson.

Back to Bessel processes.

Recall that we saw that if $X_t = |B_t|$,
B is a BM in $\mathbb{R}^n$, then

$$dX_t = d\tilde{B}_t + \frac{n-1}{2X_t}\,dt$$

We'll look instead at the squared Bessel process
$BESQ(n)$ defined by $Z_t = X_t^2$

We have
$$\begin{aligned} dZ_t &= 2X_t\,dX_t + dX_t\,dX_t \\ &= 2X_t\,d\tilde{B}_t + n\,dt \\ &= 2\sqrt{Z_t}\,d\tilde{B}_t + n\,dt \end{aligned}$$

so the squared Bessel process solves an SDE (and by the Yamada-Watanabe result, there ~~this~~ is a pathwise unique strong solution).

We can of course now extend the definition to allow non-integer values for the parameter n.

AFM

Asian options

Standard BS $S_t = S_0 e^{\sigma B_t + (r - \frac{1}{2}\sigma^2)t}$

At time T you get

$$\left(\int_0^T S_v \, \mu(dv) - K\right)^+$$

where (for example) $\mu(dt) = \frac{dt}{T}$. The aim is to calculate $\mathbb{E}\left(\int_0^T S_v \, \mu(dv) - K\right)^+$

Remarks: (i) these are traded because – they give upside optionality more cheaply than European / American calls
– they are less susceptible to price manipulation

(ii) The distribution of $\int_0^T S_v \, \mu(dv)$ is not known in closed form. Various expressions known for double Laplace transforms (in time, in strike)

What can we do?
Introduce the process $\xi_t = \left(K - \int_0^t S_v \, \mu(dv)\right) / S_t$.

Then if we look at

$$M_t = \mathbb{E}\left[\left(\int_0^T S_v \, \mu(dv) - K\right)^+ \Big| \mathcal{F}_t\right] =$$

$$= \mathbb{E}\left[\left(\int_t^T S_v \, \mu(dv) - \left(K - \int_0^t S_v \, \mu(dv)\right)\right)^+ \Big| \mathcal{F}_t\right]$$

$$= S_t \, \mathbb{E}\left[\left(\int_t^T \underbrace{\frac{S_v}{S_t}}_{\text{std. BS asset}} \mu(dv) - \xi_t\right)^+ \Big| \mathcal{F}_t\right]$$

$$= S_t \, \varphi(t, \xi_t)$$

which is a martingale

Notice that if $\mu(dt) = \rho_t\, dt$, $\rho_t \geq 0$

$$d\xi_t = -\rho_t\, dt + \left(K - \int_0^t S_u\, \mu(du)\right)\frac{1}{S_t}\left(-\sigma\, dB_t + (\sigma^2 - r)\,dt\right)$$

$$= -\rho_t\, dt + \xi_t\left(-\sigma\, dB_t + (\sigma^2 - r)\,dt\right)$$

Now we deal with the martingale

$$dM_t = S_t\left\{\varphi_t\, dt + \varphi_\xi\, d\xi + \tfrac{1}{2}\varphi_{\xi\xi}\, d\xi\, d\xi\right\}$$

$$+ \varphi S(\sigma\, dB + r\,dt) - \sigma^2 S \xi\, \varphi_\xi\, dt$$

leave out loc. mg.

$$\doteq S_t\left(\varphi_t + \varphi_\xi\left(-\rho + \xi(\sigma^2 - r)\right) + \tfrac{1}{2}\sigma^2\xi^2\varphi_{\xi\xi}\right)$$

$$+ r\varphi S - \sigma^2 S \xi \varphi_\xi \,]\, dt$$

$$= S_t\left\{\varphi_t - (\rho + r\xi)\varphi_\xi + \tfrac{1}{2}\sigma^2\xi^2\varphi_{\xi\xi} + r\varphi\right\}$$

We have a PDE for φ which has boundary cds.

$$\varphi(T, \xi) = \xi^-, \quad \text{also} \quad \varphi(t, 0) = \mathbb{E}\int_t^T \frac{S_u}{S_t}\rho_u\, du$$

$$= \int_t^T e^{r(u-t)}\rho_u\, du$$

$\rightarrow$ can be solved numerically.

<u>Approximations</u>: We're trying to calculate

$\mathbb{E}Y$, where $Y = \left(\int_0^T S_u \rho_u\, du - K\right)^+ \equiv X^+$

which we can do by first conditioning on some auxiliary r.v. Z.

$$\mathbb{E}\left(\mathbb{E}X^+ \mid Z\right) \geq \mathbb{E}\left[\mathbb{E}(X \mid Z)^+\right]$$ So if we were to use

$$Z = \int_0^T \log S_u\, \mu(du)$$