L15 $\therefore\ \mathbb{E} S_h = e^{rh} = pe^{\sigma} + qe^{-\sigma}$ $\qquad (q = 1-p)$

$$\mathbb{E} S_h^2 = e^{2rh+\sigma^2 h} = pe^{2\sigma} + qe^{-2\sigma}$$

$$A = e^{rh} = p(e^{\sigma}-e^{-\sigma}) + e^{-\sigma}$$

$$B = e^{2rh+\sigma^2 h} = p(e^{2\sigma}-e^{-2\sigma}) + \cancel{q}\, e^{-2\sigma}$$

$$A = px + qx^{-1} \qquad x = e^{\sigma} > 1$$

$$B = px^2 + qx^{-2}$$

Use the first $A = p(x - x^{-1}) + x^{-1}$

$$p = \frac{(A - x^{-1})}{x - x^{-1}}$$

So $$B = \frac{A - x^{-1}}{x - x^{-1}}(x^2 - x^{-2}) + x^{-2}$$

$$= (A - x^{-1})(x + x^{-1}) + x^{-2}$$

$$\cancel{Bx^2 = (Ax-1)(x^2+1) + 1}$$

$$\cancel{= Ax^3}$$

$$\Rightarrow (1+B)x = A(x^2+1)$$

$$\Rightarrow Ax^2 - (B+1)x + A = 0$$

$$x = \frac{B+1 \pm \sqrt{(B+1)^2 - 4A^2}}{2A}$$

Notice $B > A^2$. The larger root > 0, quadratic positive at $x=1$ and product of roots $= 1$. So want the the larger root

$$x = e^{\sigma} = \frac{B+1 + \sqrt{(B+1)^2 - 4A^2}}{2A}$$

How does this vary with h?

Both A, B close to 1 for small h.

$$A \approx 1 + rh$$
$$B \approx 1 + (2r + \sigma^2)h$$

So
$$(B+1)^2 - 4A^2 \doteq (2 + (2r + \sigma^2)h)^2 - 4(1 + rh)^2$$
$$= 4(2r + \sigma^2)h - 8rh + O(h^2)$$
$$= 4\sigma^2 h + O(h^2)$$

Thus
$$x = e^{\delta} \approx \frac{2 + \sqrt{4\sigma^2 h}}{2} = 1 + \sigma\sqrt{h}$$

So $\boxed{\delta \approx \sqrt{\sigma^2 h}}$. So here spacial steps much bigger than time steps.

This illustrates the major drawback of binomial pricing: the time step has to be much smaller than the space step.

If $T = 1$, you want $\Delta \approx 10^{-3}$, we need $\sim 10^6$ time steps ... slow, big.

It is much better to use a PDE method. Crank-Nicolson is the method of choice when it can be used

- not so good for higher-dimensional problems ...

For example, we have log-Brownian stocks, $S^1, \ldots, S^d$ and when we stop we receive

$$\left(K - \min_{1 \le i \le d}\{S^i_t\}\right)^+.$$

LIG

What more can we dig out here?

We can actually solve the infinite horizon problem explicitly. How? We expect to find that

$$V(S) = \sup_{0 \le \tau} E\left(e^{-r\tau}(K - S_\tau)^+ \mid S_0 = S\right)$$

will solve some ODE, and that $V(S^*) = K - S^*$ for some critical value S^* at which we will exercise the option.

AEM

L16 What we shall have is that $(\tau_a \equiv \inf\{u : S_u \le a\})$

$$M_t \equiv e^{-r(t\wedge\tau_a)} V_a(S_{t\wedge\tau_a}) = \mathbb{E}[e^{-r\tau_a}(K - S_{\tau_a})^+ | \mathcal{F}_t]$$

is a martingale, and $V_0(s) = (K-s)^+$

Now we do Itô on M_t,

$$dM_t = -re^{-rt} V_a(S_t)\,dt + e^{-rt}\left\{V_a' \,dS + \tfrac{1}{2} V_a'' \,dS\,dS\right\}$$

(where $dS = S(\sigma dB + r dt)$) $(t < \tau_a)$

$$= (\text{something})\,dB + e^{-rt}dt\underbrace{\left\{-rV_a + rSV_a' + \tfrac{1}{2}\sigma^2 S^2 V_a''\right\}}_{\text{vanishes, since martingale.}}$$

So we have to solve

$$\boxed{\tfrac{1}{2}\sigma^2 S^2 V_a'' + rSV_a' - rV_a = 0}$$

There 2 LI solutions of form S^θ, where θ solves:

$$\tfrac{1}{2}\sigma^2\theta(\theta-1) + r\theta - r = 0$$

which has roots $\theta = 1$, the other is $\boxed{\theta = -\frac{2r}{\sigma^2}}$

$e^{-rt}S_t$ a martingale ✓

So have $V_a(S) = AS + BS^{-2r/\sigma^2}$

Clearly $V_a(S) \to 0$ as $S \to \infty \Rightarrow A = 0$

We also have boundary condition

$$V_a(a) = Ba^{-\alpha} = K - a \quad (\alpha = 2r/\sigma^2)$$

$(a<K)$

$$\Rightarrow \boxed{B = (K-a)a^\alpha}$$

$\Rightarrow V_a(S) = (K-a)a^\alpha S^{-\alpha}$ is the value at S for this stopping rule.

Now we maximise over a (take logs first)

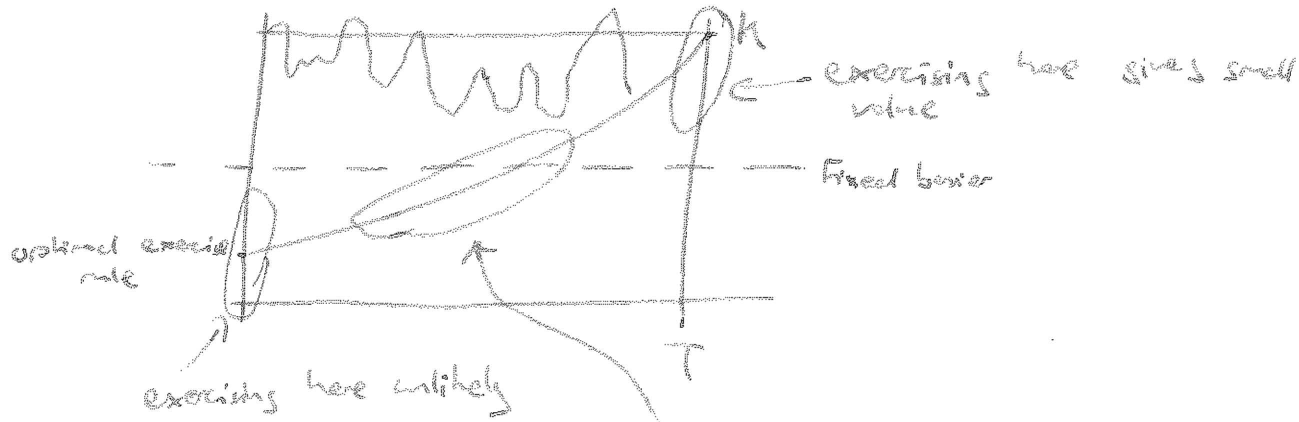
$$\Rightarrow \frac{1}{K-a} = \frac{\alpha}{a} \Rightarrow \boxed{a = \frac{\alpha K}{1+\alpha}}, \quad \alpha = \frac{2r}{\sigma^2}$$

a is the optimal barrier, ~~[illegible]~~ unsurprisingly, $a \propto K$, as it has to be by scaling.

So we can easily calculate this, as a check and as a guide: the value we obtain for infinite horizon must be $\geq$ value for finite horizon, and exercise boundary must always be $\geq \frac{\alpha K}{1+\alpha}$.

?! What if we try to use a constant boundary with a finite horizon?

This was tried by Bjerksund + Stensland in 1992, and they showed that with the best choice of the fixed barrier, the value was very nearly optimal for a wide range of realistic parameter values.



Likely to exercise in the middle, where exercise boundary is approximately constant.

<u>Exercise</u> Suppose we propose to stop at

$$\tau \equiv \inf\{t : \log S_t \leq a + bt\} \wedge T$$

(i.e. add a slope to Bj+St idea)

Find an expression for the value of the option if we use this rule. (Difficult)

<u>A different approach</u>

In a classical optimal stopping problem, we get reward Z_τ when we stop at stopping time $\tau \leq T$.

AFM

L16 **Theorem** The value of the optimal stopping problem (assuming $\sup|Z_t| \in \mathcal{L}^p$ for some $p>1$) is

$$Y_0^* = \sup_{0\le\tau\le T} \mathbb{E}(Z_\tau)$$
$$= \min_{M\in\mathcal{M}_0} \mathbb{E}\Big[\sup_{0\le t\le T}(Z_t - M_t)\Big]$$

where $\mathcal{M}_0$ is the class of UI martingales, $M_0 = 0$.

Proof $\forall M \in \mathcal{M}_0$, stopping time τ, $\mathbb{E}M_\tau = 0$

so $\mathbb{E}Z_\tau = \mathbb{E}[Z_\tau - M_\tau]$

$$\le \mathbb{E}\Big[\sup_{0\le t\le T}(Z_t - M_t)\Big]$$

so $\sup_{0\le\tau\le T} \mathbb{E}[Z_\tau] \le \mathbb{E}\Big[\sup_{0\le t\le T}(Z_t - M_t)\Big]$

since RHS does not depend on τ

Hence $\sup_{0\le\tau\le T} \mathbb{E}[Z_\tau] \le \inf_{M\in\mathcal{M}_0} \mathbb{E}\Big[\sup_{0\le t\le T}(Z_t - M_t)\Big]$

since LHS does not depend on M

For the converse, we will use a result from optimal stopping theory.
The process
$$Y_t^* = \operatorname{ess\,sup}_{t\le\tau\le T} \mathbb{E}(Z_\tau \mid \mathcal{F}_t)$$
is a supermartingale ($\operatorname{ess\,sup} = \sup\{\text{w.p. } 1\}$), and has Doob-Meyer decomposition

$$Y_t^* = Y_0^* + M_t^* - A_t^*$$

where A^* is a previsible increasing process, and M^* is in $\mathcal{M}_0$.

Now we get

$$\sup_{0\le t\le T}(Z_t - M_t^*) \le \sup_{0\le t\le T}(Y_t^* - M_t^*)$$
(by optimality of Y^*)
$$= \sup_{0\le t\le T}(Y_0^* - A_t^*)$$
$$= Y_0^*, \text{ since } A^* \text{ increasing.}$$

Therefore, $\mathbb{E}\left[\sup_{0\le t\le T}(Z_t - M_t^*)\right] \le Y_0^*$

But $Y_0^* = \sup_{\tau \in \mathcal{T}} \mathbb{E}\, Z_\tau \le \mathbb{E}\left[\sup_{0\le t\le T}(Z_t - M_t^*)\right] \le Y_0^*$

$\Rightarrow$ have equality

$\square$

Why might this be useful?

- offers a ~~simple~~ simulation route to American option pricing!

(since $\sup_{0\le t\le T}(Z_t - M_t)$ does not depend on stopping rule, it is 'easier' to find a good martingale than a good stopping rule)