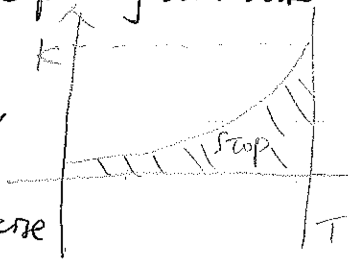
The American put

The American put option allows the holder to sell one unit of stock for strike price K at any (stopping) time. $\tau \le T$.

This must be worth at least the BS price of a European put.

What we expect to find is that the optimal exercise policy will looks like this.

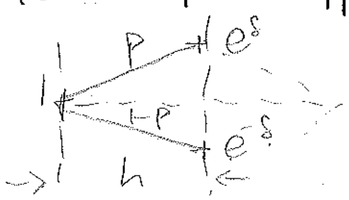
The question is to determine the exercise boundary, which has to looks like $:\longrightarrow$



* There's no closed-form expression for the exercise boundary, or for the price.
* can't simulate here ... because we don't know where we should exercise the option !!
* Direct calculation by discretizing the problem to a binomial tree then solve by dynamic programming
* PDE method ?
* Approximation ...

How would the discretization work?

To attempt to approximate the dynamics of the stock price, we could try



We've given σ, r, h and we want to choose δ, p so as to match the first two moments of the stock evolution.

$$\mathbb{E}S_h = e^{rh} \qquad \mathbb{E}S_h^2 = \mathbb{E}e^{2(\sigma B_h + (r-\frac{1}{2}\sigma^2)h)} = e^{2\sigma^2 h + 2(r-\frac{1}{2}\sigma^2)h}$$

$$= pe^{\delta} + qe^{-\delta} \qquad\qquad = e^{2rh+\sigma^2 h}$$

$$= p(e^{\delta} - e^{-\delta}) + e^{-\delta} \qquad\qquad = pe^{2\delta} + qe^{-2\delta}$$

$$p(e^{\delta}+e^{-\delta}) + (q-p)e^{-\delta} \qquad\qquad = p(e^{2\delta} - e^{-2\delta}) + e^{-2\delta}$$

we have with $e^{\delta} = x > 1$

$$\begin{cases} A = px + qx^{-1} \\ B = px^2 + qx^{-2} \end{cases}$$

Use the first: $A = p(x - x^{-1}) + x^{-1}$

$$\Rightarrow \quad p = \frac{A - x^{-1}}{x - x^{-1}}$$

The second gives

$$B = p(x^2 - x^{-2}) + x^{-2}$$

so $$B = \frac{A - x^{-1}}{x - x^{-1}}(x^2 - x^{-2}) + x^{-2}$$

$$= (A - x^{-1})(x + x^{-1}) + x^{-2}$$

$$A = p(x - x^{-1}) + x^{-1} \qquad \times \ (x + x^{-1})$$

$$A(x + x^{-1}) = p(x^2 - x^{-2}) + (1 + x^{-2})$$

$$= B - x^{-2} + (1 + x^{-2})$$

We get $\quad A(x^2+1) = (B+1)x$

$$\Rightarrow \quad Ax^2 - (B+1)x + A = 0 \Rightarrow \quad x = \frac{(B+1) \pm \sqrt{(B+1)^2 - 4A^2}}{2A}$$

Notice that $B > A^2$. The larger root is clearly >0, the quadratic is positive at $x=1$, and the product of the root is 1, so want the larger root.

$$x = e^{\delta} = \frac{(B+1) + \sqrt{(B+1)^2 - 4A^2}}{2A}$$

How does this vary with h? Both A, B are close to 1, so

$$A \approx 1 + rh \qquad B \approx 1 + (2r + \sigma^2)h$$

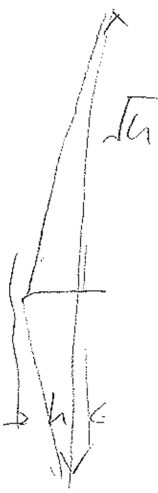
so $$(B+1)^2 - 4A^2 = (2 + (2r+\sigma^2)h)^2 - 4(1+rh)^2$$

$$= 4(2r+\sigma^2)h - 8rh + O(h^2)$$

$$= 4\sigma^2 h + O(h^2)$$

Thus $$x = e^{\delta} \approx \frac{2 + \sqrt{4\sigma^2 h}}{2} = 1 + \sqrt{\sigma^2 h}$$

so $$\boxed{\delta \approx \sqrt{\sigma^2 h}}$$



This illustrates the major drawback of binomial pricing; the time-step has to be much smaller than the space step, If $T=1$, and you want $\Delta x \approx 10^{-3}$ we need $\sim 10^6$ time steps ... slow! Big!

Much better is to use a PDE method; Crank-Nicolson is the method of choice when it can be used.

--- not so good for higher-dimensional problems ...

for example, we have n log-Brownian stocks, $S^1, \dots, S^n$, and when we stop we receive $(K - \min_{1 \le i \le n} \{S_t^i\})^+$