What about a put option? Here the payoff is $(K-S_T)^+$

We can do this the same way as the call, or notice <u>put-call parity</u>

$$(S_T-K)^+ - (K-S_T)^+ \equiv S_T - K$$

We have $\mathbb{E}e^{-rT}(S_T-K) = S_0 - e^{-rT}K$

so the put is worth

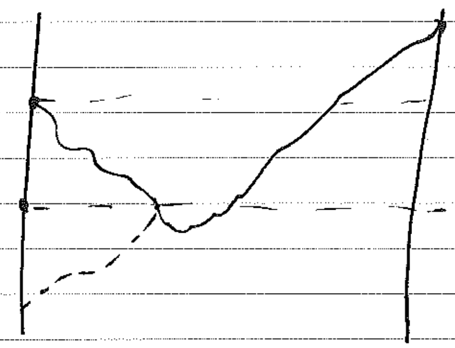
$$C(S_0, K, \sigma, r, T) - S_0 + e^{-rT}K = P(S_0, K, \sigma, r, T)$$

<u>Down-and-in Call</u>. You have a call option paying $(S_T-K)^+$ at time T only if stock has fallen to $a < S_0$ before T.

Let us suppose for simplicity $S_0 = 1$, $a = e^b \leq K$ so we have to calculate

$$\mathbb{E}^0\left[\left(e^{\sigma(B_T+\mu T)} - K\right)^+ : B_t + \mu t \leq b/\sigma \text{ for some } t \leq T\right]$$

$$\mu = (r - \tfrac{1}{2}\sigma^2)/\sigma$$



$e^{\mu B_t - \frac{1}{2}\mu^2 t}$ changes measure

$$= \mathbb{E}^0\left[(e^{\sigma B_T} - K)^+ e^{\mu B_T - \frac{1}{2}\mu^2 T} ; B_t \leq \frac{b}{\sigma} \text{ for some } t \leq T\right]$$

$$= \mathbb{E}^{\frac{2b}{\sigma}}\left(e^{\sigma B_T} - K\right)^+ e^{\mu B_T - \frac{1}{2}\mu^2 T}$$

$$= \mathbb{E}^{\frac{2b}{\sigma}}\left[(e^{\sigma B_T} - K)^+ e^{\mu(B_T - \frac{2b}{\sigma}) - \frac{1}{2}\mu^2 T}\right] e^{2\mu b/\sigma}$$

$$= \mathbb{E}^{\frac{2b}{\sigma}}\left[\left(e^{\sigma(B_T+\mu T)} - K\right)^+\right] e^{2\mu b/\sigma}$$

$\Rightarrow$ down-and-in call with strike $K \geq$ barrier $a = e^b$ is priced as

$$e^{2\mu b/\sigma} C(s_0, K, \sigma, r, T)$$

where $s_0 = e^{2b}$

Exchange option: We have two assets X, Y evolving as

$$dX = X(\sigma_X dB + \mu_X dt) \qquad \mu_X = \mu_Y = r \text{ in } \mathbb{Q}$$

~~$dY = Y(\sigma_Y d$~~

$$dY = Y(\sigma_Y(\rho dB + \rho' dB') + \mu_Y dt)$$

$|\rho| \leq 1$, $\rho' = \sqrt{1-\rho^2}$, B' is a BM independent of B.

The exchange option pays $(X_T - Y_T)^+$ at T.
How to value this?

$$\mathbb{E}\left(e^{-rT}(X_T - Y_T)^+\right) = \mathbb{E}\left[e^{-rT} X_T \left(1 - \frac{Y_T}{X_T}\right)^+\right]$$

and now we'll regard

$$e^{-rT} X_t = X_0 \exp(\sigma_X B_t - \tfrac{1}{2}\sigma^2 t)$$

as a change of measure mg. We also need to understand the dynamics Y/X:

We have $d(1/X) = -\frac{dX}{X^2} + \frac{dX\,dX}{X^3} = \frac{1}{X}(-\sigma_X dB - r\,dt + \sigma_X^2 dt)$

So: $d(Y/X) = Y d(1/X) + \frac{1}{X} dY + d(\frac{1}{X}) dY$

We have $dY = Y(\sigma_Y(\rho dB + \rho' dB') + r\,dt)$ so altogether

$$d\left(\frac{Y}{X}\right) = \frac{Y}{X}\left\{-\sigma_X dB + (\sigma_X^2 - r)dt + \sigma_Y(\rho dB + \rho' dB') + r - \sigma_X \sigma_Y \rho\, dt\right\}$$

$$= \frac{Y}{X} \left\{ \rho' \sigma_Y \, dB' - (\sigma_X - \rho\sigma_Y)(dB - \sigma_X dt) \right\}$$

The expression for the price, writing

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\sigma_X B_t - \frac{1}{2}\sigma_X^2 t}$$

becomes

$$X_0 \, E^{\tilde{\mathbb{P}}}\left(\left(1 - \frac{Y_T}{X_T}\right)^+ \right)$$

$$= X_0 \, \tilde{\mathbb{E}}\left[\left(1 - \frac{Y_0}{X_0} \cdot \exp\left(\sigma_Y \rho' \cancel{d}B_T - (\sigma_X - \rho\sigma_Y)\tilde{B}_T \underset{+\alpha T}{} \right)\right)^+ \right]$$

Where $d\tilde{B}_T \equiv dB_T - \sigma_X dt$ is a $\tilde{\mathbb{P}}$ - brownian motion

$$= X_0 \, \mathbb{E}\left(1 - \frac{Y_0}{X_0} \exp(Z)\right)^+$$

$$Z \sim N\Big(\overset{\alpha T}{\cancel{0}}, \underbrace{\{(\sigma_Y\rho')^2 + (\sigma_X - \rho\sigma_Y)^2\}}_{\sigma_Y^2 - 2\rho\sigma_X\sigma_Y + \sigma_X^2} T\Big)$$

where $\alpha = -\frac{1}{2}(\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2)$