AFM - 4th Nov

$$dS_t = S_t(\sigma dB_t + \mu_t dt)$$

measure $\mathbb{Q} \sim \mathbb{P}$, $\Lambda_t \equiv \left.\dfrac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$ Solves $\underbrace{d\Lambda_t = \Lambda_t \alpha_t dB_t}_{\cancel{\text{is mart}}}$

Then $\tilde{B}_t \equiv B_t - \int_0^t \alpha_s ds$ is a $\mathbb{Q}$-~~mart~~ Brownian Motion

($\Lambda_t \tilde{B}_t$ is a $\mathbb{P}$-mart; this shows that $\tilde{B}$ is a $\mathbb{Q}$-mart; we know that $\tilde{B}$ is continuous and has quadratic variation dt so must be a BM.)

So in our context, if we change measure to $\mathbb{Q}$ we get

$$dS_t = S_t(\sigma dB_t + \mu_t dt) = S_t(\sigma(d\tilde{B}_t + \alpha_t dt) + \mu_t dt)$$

$$= S_t(\sigma d\tilde{B}_t + (\sigma\alpha_t + \mu_t)dt)$$

we want $\mathbb{Q}$ to turn the stock price into a martingale and for this we need the drift term to vanish: ($r=0$ for now)

$$\sigma\alpha_t + \mu_t = 0$$

leaving $dS_t = S_t \sigma d\tilde{B}_t$

So this tells us that need to use the change of measure given by $\boxed{\alpha_t = -\sigma^{-1}\mu_t}$ when we have non-zero interest rate, we want ~~to~~ $e^{-rt}S_t = \tilde{S}_t$ to be a $\mathbb{Q}$ mart $d\tilde{S}_t = \tilde{S}_t(-r dt + \sigma dB_t + \mu_t dt)$

So now we would take $\alpha_t = \sigma^{-1}(r - \mu_t)$

<u>In summary</u>: when we put ourselves into the equivalent martingale measure

$$\boxed{dS_t = S_t \{ \sigma d\tilde{B}_t + r\,dt \}}$$ Black-Scholes

we can solve this to give $S_t = S_0 \exp\{\sigma B_t + (r - \frac{1}{2}\sigma^2)t\}$

<u>Note:</u>

Suppose we take some asset return data and we demonstrate that the returns are not iid Gaussian. Does this invalidate the Black Schole model? NO - BS model is about the $\mathbb{Q}$ dynamics of the stock price S, real world is $\mathbb{P}$.

To invalidate BS, we would have to look at prices of traded assets such as options.

(So we could ask for the time-0 price of a call option with strike K, expiry T

$$C(S_0, K, \sigma, r, t) \equiv \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+\right])$$

Do observed option prices fit the formula?

know $S_0, K, r, T, \ldots \sigma?$
most difficult of 5 parameters to identify

- estimation is erroneous & backward looking

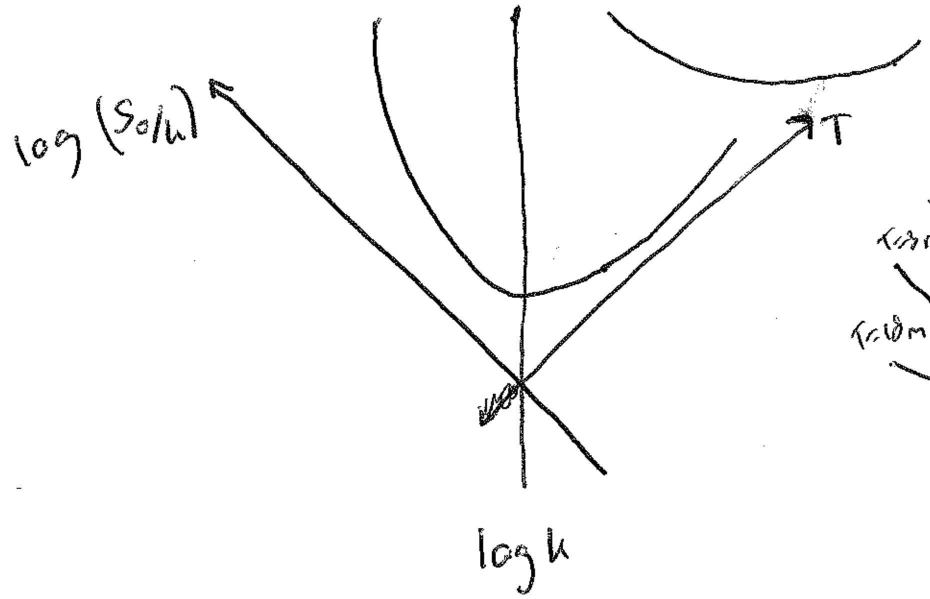
Never the less, if BS story is true, if we look at the prices of options with strikes $K_1, \ldots, K_n$; expiries $T_1, \ldots, T_m$, and try to find one value of σ which prices them all... but it does not work.

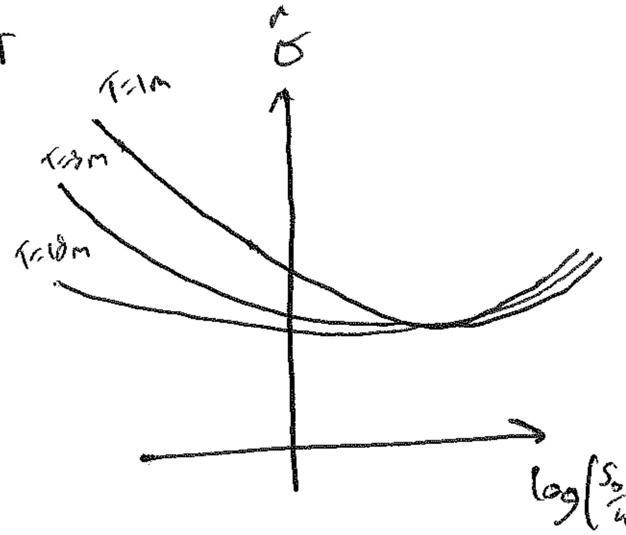
- So we shouldn't trust BS and its conclusions?
- In the industry, people work with concept of <u>implied vol</u>

you observe that a call option with strike k, expiry T is trading for Y, so you say the implied vol $\hat{\sigma}(T, k, r, S_0)$ is the value of σ such that the B.S. formula is true.

$$Y = C(S_0, k, \sigma, r, T)$$

The implied vol surface typically looks like





BS option pricing formula: $S_t = S_0 \exp(\sigma B_t + (r - \frac{1}{2}\sigma^2)t)$

$$\mathbb{E}\left[(S_T - k)^+ e^{-rT}\right]$$

$$= e^{-rT} \int \left(S_0 e^{\sigma\sqrt{T}x + (r - \frac{1}{2}\sigma^2)T} - k\right)^+ e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-rT} \int_a^\infty \left(S_0 e^{\sigma\sqrt{T}x + (r - \frac{1}{2}\sigma^2)T} - k\right) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

where $a = \frac{1}{\sigma\sqrt{T}}\left\{\log k/S_0 - (r - \frac{1}{2}\sigma^2)T\right\}$

$$= e^{-rT}\int_a^\infty \left\{ S_0 \exp\left(-(x-\sigma\sqrt{T})^2/2 - rT\right) - k e^{-x^2/2} \right\} \frac{dx}{\sqrt{2\pi}}$$

$$= S_0 \bar{\Phi}(a - \sigma\sqrt{T}) - k e^{-rt} \bar{\Phi}(a)$$

where $\bar{\Phi}(y) = \int_y^\infty e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$