B_t a BM in $\mathbb{R}^d$, $X_t = |B_t|$; we saw

$$dX_t = d\tilde{B}_t + \frac{d-1}{2X_t}\,dt$$

and $\varphi(X_t)$ is a local martingale if

$$\varphi(x) = \begin{cases} \log x & (d=2) \\ x^{2-d} & (d\geq 3) \end{cases}$$

In dimension $d \geq 3$, we can conclude for $0 < \varepsilon \leq X_0 \leq a$ that

$$P^{X_0}(\text{hit } \varepsilon \text{ before } a) = \frac{\varphi(a) - \varphi(X_0)}{\varphi(a) - \varphi(\varepsilon)}$$

$$= (a^{2-d} - X_0^{2-d}) / (a^{2-d} - \varepsilon^{2-d})$$

So we find that for $d \geq 3$

$$P^{X_0}(\text{hit } \varepsilon) = (\varepsilon / X_0)^{d-2} < 1$$

So BM in dimension $d \geq 3$ is transient

__Remark:__ The process X_t is called a d-dimensional Bessel process, BES(d). The 3-dimensional Bessel process is particularly interesting. Notice that:

(i) $1/X_t$ is a local martingale : $d(1/X_t) = (-1/X_t^2)\,d\tilde{B}_t$
but not a martingale
– can calculate $E(1/X_t) \to 0 \qquad (t \to \infty)$

(ii) Nevertheless (assuming $B_0 \sim N(0, I)$) we have $1/X_t \in L^2 \ \forall t$

What has this to do with math finance?
In discrete time, FTAP says
no arbitrage $\iff$ $\exists$ equivalent martingale measure Q :
S_t / S_t^0 is a Q-martingale

In continuous time, nothing so complete happens
(Delbaen – Schachermayer)

$$NFLVR \iff \exists\ \sigma\text{-martingale measure } Q$$

For us, the axiomatic starting point will be that <u>there exists an equivalent martingale measure</u>

FLVR: $\exists (\theta_n)$ s.t.

(i) $(\theta_n \cdot S)_t \geq -1 \quad \forall 0 \leq t \leq 1$

(ii) $\exists \varepsilon > 0$ s.t. $\mathbb{P}((\theta_n \cdot S)_1 > \varepsilon) > \varepsilon$

(iii) $\forall \delta > 0$, $\mathbb{P}((\theta_n \cdot S)_1 < -\delta) \to 0$

Let's straight away restrict attention towards based on Brownian motion. Suppose we've got a single stock, price at time t is S_t where

$$dS_t = S_t(\sigma dB_t + \mu_t dt)$$

where $\sigma > 0$ is constant, and μ is bounded previsible

Our axiomatic strategy point says there is some $\mathbb{Q}$ equivalent to $\mathbb{P}$ s.t. S is a $\mathbb{Q}$-martingale

(so let's for simplicity take $r=0$, or equivalently that we're talking about $e^{-rt}S_t \equiv \tilde{S}_t$)

What does an equivalent mart measure look like?

What does an equivalent measure look like?

Consider $\Lambda_t = \left.\dfrac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$

where $\mathbb{Q} \sim \mathbb{P}$. Then Λ is a $\mathbb{P}$-martingale, strictly positive, $\Lambda_0 = 1$. If we now assume that Λ is L^2-bounded on each $[0, N]$, then by the Brownian integral Representation ~~of~~ Theorem, there exists some previsible H s.t.

$$\mathbb{E}\left(\int_0^K H_s^2\, ds\right) < \infty \quad \forall K$$

such that $\Lambda_t = 1 + \int_0^t H_s\, dB_s$

Thus we have $d\Lambda_t = \Lambda_t \alpha_t\, dB_t$ where $\alpha_t = H_t / \Lambda_t$

So we may deduce that

$M_t = \Lambda_t (B_t - \int_0^t \alpha_s\, ds)$ is a $\mathbb{P}$-martingale

because $dM_t = (B_t - \int_0^t \alpha_s ds)\, d\Lambda_t + \Lambda_t (dB_t - \cancel{\alpha_t dt})$

$\qquad + \cancel{\alpha_t \Lambda_t dt}$

$\qquad = \{ - \} dB_t$

$\tilde{B}_t$

Here (Exercise!) deduce that $B_t - \int_0^t \alpha_s ds$ is a Q-martingale, and that it is a Q-Brownian motion

$$dS_t = S_t(\sigma(d\tilde{B} + \alpha dt) + \mu_t dt)$$