Stochastic differential equations

What could we do with stochastic differential equations:

$$dX_t = \sigma(t, X_t)\, dB_t + \mu(t, X_t)\, dt. \qquad \text{Lipschitz}$$
$$X_t = x_0$$

Just as in classical ODE theory, this SDE will have an unique solution if for each $T > 0$, there exists $G_T < \infty$ s.t.

$\forall x, y \in \mathbb{R}^n$ $\quad |\sigma(t,x) - \sigma(t,y)| + |\mu(t,x) - \mu(t,y)| \leq G_T |x-y|$

$\forall t \in [0,T]$ $\quad |\sigma(t,0)| + |\mu(t,0)| \leq G_T$

The proof is constructive: start with $X_t^{(0)} \equiv x_0$ and define recursively

$$X_t^{(n+1)} = x_0 + \int_0^t \sigma(s, X_s^{(n)})\, dB_s + \int_0^t \mu(s, X_s^{(n)})\, ds.$$

In one dimension, we can do better, we still require μ to be Lipschitz, but we can find an unique sol^n provided

$$|\sigma(t,x) - \sigma(t,y)| \leq G\, |x-y|^{\alpha}.$$

for all x, y $\forall t \in [0,T]$ where $\alpha \in [\frac{1}{2}, 1]$ is fixed

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$$dX_t^i = \sigma_{ij}(X_t)\,dB_t^j + \mu^i(X_t)\,dt$$

There is an unique sol^n (X_t) for any $X_0 = x_0$ if the σ, μ are Lipschitz.

Remark that in 1-dimension, Hölder (α) for σ for some $\alpha \geq \frac{1}{2}$ is sufficient

eg(1) $dX_t = \sigma X_t\,dB_t + \mu X_t\,dt$.

This we can solve explicitly. If we look at $Y_t \equiv \log X_t$, we have $C^{1,2}$.

we see $dY_t = \frac{1}{X_t}dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)dX_t\,dX_t$.

Itô !

$$= \sigma dB_t + \mu dt - \tfrac{1}{2}\sigma^2\,dt.$$

$$\Rightarrow Y_t = \log X_t = \log x_0 + \sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t.$$

$$\Rightarrow X_t = x_0 \exp\left(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\right)$$

Notice if $\mu = 0$, the sol^n X_t is the Brownian exponential martingale $\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ & the SDE says X_t is a (local) martingale.

eg(2) The Ornstein - Uhlenbeck (OU) process solves

$$dX_t = \sigma\,dB_t - \lambda X_t\,dt.$$

This is also amenable to exact sol^n:

Consider $d(e^{\lambda t}X_t) = \lambda e^{\lambda t}dt\,X_t + e^{\lambda t}dX_t + 0$

$$= e^{\lambda t}(dX_t + \lambda X_t\,dt)$$

$$= \sigma e^{\lambda t}\,dB_t$$

Hence $\boxed{e^{\lambda t}X_t - X_0 = \int_0^t \sigma e^{\lambda s}\,dB_s}$

Thus $\boxed{X_t = e^{-\lambda t}\left(X_0 + \underbrace{\int_0^t \sigma e^{\lambda s}\,dB_s}_{\text{zero mean Gaussian R.V. with variance}}\right)}$

$$E\left(\sigma e^{-\lambda t}\int_0^t e^{\lambda s}dB_s\right) = \sigma^2 e^{-2\lambda t}E\left[\int_0^t e^{2\lambda s}ds\right] = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t})$$

Exercise: If $X_t = \int_0^t f(s)\,dB_s$, where $f \in L^2(\mathbb{R}^+)$, show that
(use Itô)

$$\exp\left(iQ X_t + \frac{Q^2}{2}\int_0^t f(s)^2\,ds\right)$$ is a martingale for any $Q \in \mathbb{R}$.

& deduce that $X_t \sim N(0, \int_0^t f(s)^2\,ds)$

Remark: Assuming $\lambda > 0$, we find that as $t \to \infty$.
$\Leftrightarrow \xrightarrow{d}$ $X_t \Rightarrow N(0, \sigma^2/2\lambda)$. This is an ergodic diffusion & the simplest example

Eg(3) Suppose B_t is a BM in $\mathbb{R}^d$, $d \geq 2$, and let's consider $X_t = |B_t|$. To avoid singularity questions, let's suppose $B_0 \neq 0$. We have the f^n $f(x) = \sqrt{\sum x_i^2}$ is C^∞ away from 0.

We can apply Itô to X_t and get

$(X_t = f(B_t))$
$$dX_t = D_i f(B_t)\,dB_t^i + \frac{1}{2}\Delta f(B_t)\,dt$$
$$= \frac{B_t^i}{|B_t|}\,dB_t^i + \frac{1}{2}\,\frac{d-1}{|X_t|}\,dt.$$

We see that X_t solves
$$dX_t = d\hat{B}_t + \frac{d-1}{2X_t}\,dt$$

where $\hat{B}$ is the first piece, $B_t^i\,dB_t^i/|B_t|$, a cts local martingale, whose quadratic variation is
$$d\hat{B}\cdot d\hat{B} = \sum_{i=1}^d \frac{(B_t^i)^2}{|B_t^i|^2}\,dt = dt$$

So $\hat{B}_t$ is a cts local martingale, $\hat{B}_t^2 - t$ is a cts local martingale, so (Lévy) $\hat{B}$ is a Brownian motion

Can we find some f^n φ s.t. $\varphi(X_t)$ is a local martingale? We would need

$$d\varphi(X_t) = \varphi'(X_t)\,dX_t + \tfrac{1}{2}\varphi''(X_t)\,dX_t\,dX_t$$
$$= \varphi'(X_t)\,d\hat{B}_t + \underbrace{\left\{\tfrac{1}{2}\varphi''(X_t) + \tfrac{d-1}{2X_t}\varphi'(X_t)\right\}dt}_{=0 \Leftarrow \text{ want a martingale}}$$

So we find φ must solve.

$$\varphi''(x) + \frac{d-1}{2x}\varphi'(x) = 0.$$

$$\Rightarrow \quad \varphi'(x) = \cancel{\text{const}}\; x^{-d+1}$$
$$\Rightarrow \quad \varphi(x) = \begin{cases} \log x & \text{if } d=2 \\ x^{2-d} & \text{if } d \geq 3 \end{cases}$$

up to some constants

From this, we can deduce information about the reccurence / transience of BM in $\mathbb{R}^d$ ~~fast~~

Let $\tau = \inf\{t : |B_t| = \varepsilon \text{ or } |B_t| = a\}$

Then $\varphi(X(t \wedge \tau))$ is bounded, also a local martingale $\therefore$ A martingale.

We can use OST to deduce that

$$\varphi(x_0) = \mathbb{E}[\varphi(X_\tau)]$$
$$= \mathbb{P}^{x_0}(\text{hit } \varepsilon \text{ before } a)\,\varphi(\varepsilon) + \mathbb{P}^{x_0}(\text{hit } a \text{ before } \varepsilon)\,\varphi(a)$$
$$= \mathbb{P}^{x_0}(\text{hit } \varepsilon \text{ before } a)(\varphi(\varepsilon) - \varphi(a)) + \varphi(a)$$
$$\Rightarrow \quad \mathbb{P}^{x_0}(\text{hit } \varepsilon \text{ before } a) = \frac{\varphi(x_0) - \varphi(a)}{\varphi(\varepsilon) - \varphi(a)}$$

So what we get is in $d=2$,

$$\mathbb{P}^{x_0}(\text{hit } \varepsilon \text{ before } a) = \frac{\log(x/a)}{\log(\varepsilon/a)}$$

If we fix a & let $\varepsilon \downarrow 0$, we get

$$\mathbb{P}^{x_0}(\text{hit } \varepsilon \text{ before } a) \longrightarrow 0$$

Hence probability (hit 0 before we hit a) $= 0$.

$\therefore \mathbb{P}(\text{hit } 0 \text{ at some time}) = 0.$

But If we fix $\varepsilon > 0$ & let $a \to \infty$.

$\mathbb{P}^{x_0}$ (hit ε before a) $\longrightarrow 1$.

$\Rightarrow$ $\mathbb{P}^{x_0}$ (hit ε at some time) $= 1$.