Advanced Financial Models — 28th October

$$df(t,B_t) = \boxed{f'(t,B_t)\,dB_t} + \left\{\dot{f} + \tfrac{1}{2}f''\right\}(t,B_t)\,dt$$

Extending the stochastic Integral

We've seen how Itô's formula works for C^3_{bounded} functions of (t, B_t), but we really only need f to be $C^{1,2}$. Then the formula makes sense. But theres a catch...

The term $\int_0^t f'(s,B_s)\,dB_s$ is generally not a martingale. Only a local martingale.

A process M is a local martingale if there exists Stopping times $T_n \uparrow \infty$ such that $M^{T_n}_t = M_{t\wedge T_n}$ is a mart.

For example;

$X_t \equiv \int_0^t \exp(B_s^4)\,dB_s$ is a local martingale, but not a martingale; we could take

$$T_n = n \wedge \min\{t : |B_t| \geq n\}$$

Then when we take $X(t\wedge T_n) = \int_0^{t\wedge T_n} \exp(B_s^4)\,dB_s$ the integrand $\exp(B_s^4)\,I_{\{s \leq T_n\}}$ is bounded, and zero off the time interval $[0,n]$.

So the integrand is in $\bar{\mathcal{S}}$, and the stochastic integral is an L^2 bounded martingale.

But without localisation

$$\mathbb{E}\left[\int_0^2 e^{2B_s^4}\,ds\right] = +\infty$$

So the original process has no nice integrability properties.

This frees us from the need to constantly check the integrability of the integrand.

Another extension is to multidimensional B.M.

$$B_t = (B_t^1, \ldots, B_t^n)$$

where the B^j are independent standard Brownian Motions.

The rules of the calculus are

$$\boxed{dt^2 = 0 \quad dt\, dB_t = 0 \quad dB_t^i\, dB_t^j = \delta_{ij}\, dt}$$

and now

$$d(f(t,B_t)) = \dot{f}\, dt + \sum_{i=1}^{n} (D_i f)(t,B_t)\, dB_t^i$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} D_{ij} f(t,B_t)\, dB_t^i dB_t^j$$

$$= \dot{f}\, dt + (D_i f)(t,B_t)\, dB_t^i$$

$$+ \frac{1}{2} \Delta f(t,B_t)\, dt$$

Summation convention

<u>Next extension</u>

If $X_t = x + \int_0^t H_s\, dB_s + \int_0^t N_s\, ds$ then

$$\cancel{}\; d(f(t,X_t)) = \dot{f}\, dt + f'(t,X_t)\, dX_t + \frac{1}{2} f''(t,X_t)\, dX_t\, dX_t$$

$$= \dot{f}\, dt + f'(t,X_t)(H_t\, dB_t + N_t\, dt) + \frac{1}{2} f''(t,X_t) H_t^2\, dt$$

<u>Multivariate</u>

adapted and pathwise integrable $t \mapsto N_t^i(\omega)$

$$dX^i = H^{ij}\, dB^j + N^i\, dt$$

locally in $L^2(\tilde{\Omega}, \text{Leb} \times \mathbb{P})$

then
$$d(f(t,X_t)) = \dot{f}\, dt + (D_i f)(t,X_t)\, dX_t^i + \frac{1}{2}(D_{ij} f)(t,X_t)\, dX_t^i \cdot dX_t^j$$

dropping t subscripts

$$= \dot{f}\, dt + D_i f(t,X)\left\{ H^{ik}\, dB^k + N^i\, dt \right\} + \frac{1}{2}(D_{ij} f)(t,X_t)$$

$$[dX^i dX^j = H^{jl} dB^l H^{ik} dB^k = H^{jl} H^{il} dt]$$

<u>Special Case</u> $f(x,y) = xy$

(Integration by parts formula)

Suppose $dX = H\,dB + N\,dt$
$dY = G\,dB + v\,dt$

$$d(XY) = Y\,dX + X\,dY + 1 \cdot dX\,dY$$

$$= Y\,dX + X\,dY + HG\,dt$$

e.g.

$$dX^2 = 2X\,dX + dX\,dX$$

<u>Stochastic differential Equations</u>

What could we do with S.d.e's;

$$dX_t^i = \sigma^{ij}(t, X_t)\, dB_t^j + N^i(t, X_t)\, dt$$

"Lipshitz + Picard Iteration"

$$X_t = x_0$$

Just as in classical ODE theory, this SDE will have a unique solution if for each $T > 0$ there exists $C_T < \infty$ such that

$$|\sigma(t,x) - \sigma(t,y)| + |N(t,x) - N(t,y)| \leq C_T |x - y|$$

$$|\sigma(t,0)| + |N(t,0)| \leq C_T \qquad \forall x,y \in \mathbb{R}^n \quad \forall t \in [0,T]$$

The proof is constructive; start with $X_t^{(0)} \equiv x_0$ and define recursively

$$X_t^{(n+1)} = x_0 + \int_0^t \sigma(s, X_s^{(n)})\, dB_s + \int_0^t N(s, X_s^{(n)})\, ds$$

In one dimension, we can do better; we still require μ to be Lipschitz, but we can find a unique solution

provided
$$|\sigma(t,x) - \sigma(t,y)| \leq C_T |x-y|^{\alpha}$$
for all x,y, $\forall$ $t \in [0,T]$, where $\alpha \in \left[\frac{1}{2}, 1\right]$ is fixed

(Yamada - Watanabe; Le Gall).