Advanced Financial Models

'A first course on derivative pricing'.

Prerequisites: Assume measure theory, martingales. (discrete time effectively)

[redacted] Books: Baxton-Rennie, Pliska

Notes: Mike Techranchi on course website

Contents:

0. Where do prices come from?

Q: Axiomatic basis?
Equilibrium?

An axiomatic approach:
Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space. Then we can consider a family $\pi_{st} : L^\infty(\mathcal{F}_t) \to L^\infty(\mathcal{F}_t)$

of pricing operators $(0 \le s \le t)$ which we interpret as

$\pi_{st}(Y)$ is the time-s market price for a contingent claim ($\equiv$ random var.) $Y \in L^\infty(\mathcal{F}_t)$

We'll suppose that the (π_{st}) satisfy:

(A1) each π_{st} is a bounded positive linear op.

(A2) If $Y \in L^\infty(\mathcal{F}_t)$ and $Y \geq 0$ then

$$\pi_{0t}(Y) = 0 \quad \text{iff} \quad \mathbb{P}(Y=0) = 1$$

(A3) For $0 \le s \le t \le u$, $X \in L^\infty(\mathcal{F}_t)$, $Y \in L^\infty(\mathcal{F}_u)$

$$\pi_{su}(XY) = \pi_{st}\left(X \pi_{tu}(Y)\right) \quad \leftarrow \text{look at intermediate time}$$

(A4) For each $t \geq 0$ π_{0t} is bounded monotone continuous: if $|Y_n| \le 1$, $Y_n \uparrow Y$ then $\pi_{0t}(Y_n) \uparrow \pi_{0t}(Y)$

<u>THEN</u> there exists a strictly positive process $(\zeta_t)_{t \geq 0}$ such that

$$\pi_{st}(Y) = \frac{1}{\zeta_s} \mathbb{E}\left[\zeta_t Y \mid \mathcal{F}_s\right]$$

This pricing formula is the recurrent theme of the course.

<u>Proof</u>: Fix $t > 0$ and consider

$$A \longmapsto \pi_{0t}(I_A) \qquad (A \in \mathcal{F}_t)$$

This is a measure by (A1), (A4) ← for countable additivity

We also have that the measure is absolutely continuous wrt. $\mathbb{P}$ by (A2), as

$$\mathbb{P}(A) = 0 \implies \pi_{0t}(I_A) = 0$$

By Radon-Nikodym there exists a density ζ_t such that

$$\pi_{0t}(I_A) = \int_A \zeta_t \, d\mathbb{P}$$

where ζ_t is $\mathcal{F}_t$-measurable non-negative.
By (A2) again, we have $\mathbb{P}(\zeta_t = 0) = 0$.
(implication in A2 is in both directions)
Finally we have (A3) $\mathcal{F}_t$

$$\pi_{0u}(XY) = \int XY \zeta_u \, d\mathbb{P}$$

$$= \pi_{0t}\left(X \pi_{tu}(Y)\right)$$

$$= \int X \pi_{tu}(Y) \zeta_t \, d\mathbb{P}$$

This has to be true $\forall Y \in L^\infty(\mathcal{F}_u)$, $\forall X \in L^\infty(\mathcal{F}_t)$
Hence

$$\int X \cdot Y \zeta_u \, d\mathbb{P} = \int X \, E\left(Y\zeta_u \mid \mathcal{F}_t\right) d\mathbb{P}$$

$$= \int X \cdot \pi_{tu}(Y) \cdot \zeta_t \, d\mathbb{P}$$

$$\Rightarrow \pi_{tu}(Y)\zeta_t = \mathbb{E}_t Y \zeta_u \quad \text{a.s}$$

$$\Rightarrow \pi_{tu}(Y) = \frac{1}{\zeta_t} \mathbb{E}\, Y \zeta_u \mid \mathcal{F}_t$$

Remarks: We'll see that equilibrium pricing and APT also lead to $\boxed{}$ ... so why do them?
... the devil is in the domain: we assumed
$\pi_{st} : L^\infty(\mathcal{F}_t) \to L^\infty(\mathcal{F}_s)$
We assumed that every $\mathcal{F}_t$-measurable r.v. has a market price

... perhaps we could restrict Π_{st} to $V \subset L^\infty(\mathcal{F}_t)$?
but can we extend the linear functional Π_{0t}?
Hahn-Banach $\rightarrow$ $(L^\infty)^* \supset L^1$ is not nice.
OR $\Pi_{st} : V \subset L^p(\mathcal{F}_t) \to L^p(\mathcal{F}_s)$

but the integrability wrt. $\mathbb{P}$ is not ~~clear~~ correct
we need integrability wrt $\zeta_t \, d\mathbb{P}$
Why don't we just have
$$\Pi_{0t}(Y) = \mathbb{E}Y?$$
If we have a bank account with constant interest rate r then $\Pi_{0t}(1) = e^{-rt}$
so prices should reflect discounting.
So why not $\Pi_{0t}(Y) = \mathbb{E}e^{-rt}Y$?

It also does not get it right...