Introduction to Probability

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Example Sheet 3, Michaelmas Term 2008.

Problem 1. Let $(X_n, n \geq 0)$ be a Markov chain on S. Let $p_{ij}(n)$ be the n-step transition probabilities. Show that these number satisfy the *Chapman-Kolmogorov* equations: for all $m, n \geq 0$ and for all $i, j \in S$:

$$p_{ij}(m+n) = \sum_{k \in S} p_{ik}(m) p_{kj}(n)$$

Note that this implies in particular that the n-step transition matrix is the n^{th} power of the one-step transition matrix.

Problem 2. Let $(X_n)_{n \geq 0}$ be a homogeneous Markov chain on S with transition matrix P. Given a $k \in \mathbb{N}$, let $Z_n = X_{kn}$. Prove that $(Z_n)_{n \geq 0}$ is Markov chain with transition matrix P^k.

Problem 3. Let $(X_n, n \geq 0)$ be a Markov chain on a state space S. Let $D \subset S$ be a set of states. Let $T = \inf\{n \geq 0 : X_n \in D\}$ be the time of the first visit to D, including $T = 0$ if the chain starts in D. For $i \in S$ let $h_i = \mathbb{P}_i(T < \infty)$ be the hitting probability of D starting from i. The goal of this problem is to show that h_i is the minimal nonnegative solution to the set of equations:

$$\begin{cases} h_i = 1 & \text{if } i \in D \\ h_i = \sum_{j \in S} p_{ij} h_j & \text{else} \end{cases} \tag{1}$$

Here, *minimal solution* means that if $x_i, i \in S$ is another nonnegative solution to (1) then $x_i \geq h_i$ for all $i \in S$.

(a) Show that $h_i, i \in S$ satisfies (1).

(b) Let $x_i, i \in S$ be a nonnegative solution of (1). Show that if $i \notin D$, for any $n \geq 1$,

$$\begin{aligned} x_i &= \mathbb{P}_i(X_1 \in D) + \ldots + \mathbb{P}_i(X_1 \notin D, \ldots X_{n-1} \notin D, X_n \in D) \\ &\quad + \sum_{j_1 \notin D, \ldots j_n \notin D} p_{ij_1} p_{j_1 j_2} \ldots p_{j_{n-1} j_n} x_{j_n} \end{aligned}$$

Conclude that $x_i \geq h_i$.

Note: In practice, one can often obtain explicit expressions for h_i by solving (1). This will be illustrated in one example below.

Problem 4. (Recurrence of Simple Random Walk in 1 dimension, first solution.) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$$

for all n. Let $S_n = X_1 + \ldots + X_n$.

(a) Show that $\mathbb{P}_0(X_{2n} = 0) = \binom{2n}{n} 2^{-2n}$.

(b) Using Stirling's formula: $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$, show that $(S_n, n \geq 0)$ is recurrent.

Problem 5. (Recurrence of Simple Random Walk in 1 dimension, second solution.) Let $(S_n, n \geq 0)$ be the Markov chain of the previous problem. Let $D = \{0\}$, and let $T = \inf\{n \geq 0 : S_n = 0\}$ be the hitting time of 0. For $i \in \mathbb{Z} = \{\dots, -1, 0, 1, \}$, let $h_i = \mathbb{P}_i(T < \infty)$ be the hitting probability of zero. Show (using e.g. (1)) that for $i \geq 0$, $(h_{i+1} - h_i)$ does not depend on i. Conclude that $h_1 = h_{-1} = 1$ and that the chain is recurrent.

Optional Problem. Prove that in dimension 2, simple random walk is recurrent, while in dimension 3 or higher it is transient. That is, let $d \geq 2$. and let $X_1, X_2, \dots$ be a sequence of independent vector-valued random variables with

$$\mathbb{P}(X_n = e_i) = \mathbb{P}(X_n = -e_i) = \frac{1}{2d}$$

for all $1 \leq i \leq d$, where

$$e_i = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{d-i})$$

Let $S_n = X_1 + \cdots + X_n$. Prove that $(S_n)_{n \geq 0}$ is a transient Markov chain on $\mathbb{Z}^d$ if and only if $d \geq 3$.

[You may find the following inequality useful: If $i_1 + \ldots + i_d = dn$ then $i_1! \cdots i_d! \geq (n!)^d$.]

Problem 6. Let T_1 and T_2 be stopping times for a Markov chain $(X_n)_{n \geq 0}$ on S. Prove that each of the following are also stopping times:

1. $T = \min\{n \geq 1 : X_n = i\}$ for some fixed $i \in S$.
2. $T(\omega) = N$ for all $\omega \in \Omega$ for a fixed $N \in \mathbb{N}$.
3. $T = \min\{T_1, T_2\}$.
4. $T = \max\{T_1, T_2\}$.
5. $T = T_1 + T_2$.

Problem 7. Let $(X_n, n \geq 0)$ be an irreducible Markov chain on some state space S. For $x \in S$, let $T_x = \inf\{n \geq 1 : X_n = x\}$. For $x \in S$, let $\gamma^x = (\gamma^x_y)_{y \in S}$ be the vector defined by

$$\gamma^x_i = \mathbb{E}_x \left(\sum_{n=0}^{T_x - 1} \mathbf{1}_{\{X_n = i\}} \right).$$

Show that $(\gamma^x_i)_{i \in S}$ defines an invariant measure for every $x \in S$.

Problem 8. Let $(X_n, n \geq 0)$ be an irreducible Markov chain on some state space S.

(a) For $i, x \in S$, let $\gamma_i^x = \mathbb{E}_x\left(\sum_{n=0}^{T_x-1} \mathbf{1}_{\{X_n=i\}}\right)$ where $T_x = \inf\{n \geq 1 : X_n = x\}$. Show that $\gamma_x^x = 1$ and that

$$\gamma_i^x = \lim_{n\to\infty} \mathbb{P}_x(X_1 = i, T_x \geq 1) + \ldots + \mathbb{P}_x(X_n = i, T_x \geq n).$$

(b) Let $\lambda = (\lambda_i)_{i\in S}$ be an invariant vector such that $\lambda_x = 1$ for some $x \in S$. By using (a) and a reasoning analogous to Problem 3, show that $\lambda_i \geq \gamma_i^x$ for all $i \in S$.

(c) Show that the following are equivalent:

(i) All states are positive recurrent.

(ii) Some state is positive recurrent.

(iii) There exists an invariant distribution for the chain.

Hint: For (ii) $\Rightarrow$ (iii) one can use Problem 7. For (iii) $\Rightarrow$ (i) let $\lambda_y = \pi_y/\pi_x$ where x is such that $\pi_x > 0$. Then use (b).

(d) (optional) Show that if any of (i), (ii) or (iii) holds, then any invariant distribution π must be of the form $\pi_i = 1/\mathbb{E}_i(T_i), i \in S$. In particular if the chain is positive recurrent, then invariant distributions exist and are unique.

Problem 9. A fair die is rolled repeatedly. Let X_n be the sum of the first n throws. Show that

$$\lim_{n\to\infty} \mathbb{P}(X_n \text{ is a multiple of 13 })$$

exists and compute it. (State carefully the results you are using).

Problem 10. A flea hops randomly on the vertices of a triangle with vertices labelled 1,2, and 3, hopping to each of the other vertices with equal probability. If the flea starts at vertex 1, find the probability that after n hops the flea is back to vertex 1. (Hint: what are the eigenvalues of the transition matrix?)

A second flea also starts at vertex 1 and hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back to vertex 1?

Problem 11. Consider the second flea of Problem 10. What is the expected number of hops the flea makes before it is first back to vertex 1? What is the expected number of times the flea visits vertex 3 before first reaching vertex 2? (Assume that the vertices are labelled so that $1 \to 2 \to 3 \to 1 \ldots$ is clockwise.)