Mock Exam:

Mixing Times of Markov Chains (Lent 2017).

Duration: two hours.

Attempt no more than THREE questions.

There are FOUR questions in total.

The questions carry equal weight.
Problem 1.

(a) Define the total variation distance \( \| \mu - \nu \|_{tv} \) for probability distributions \( \mu, \nu \) on a finite set \( S \). Show that
\[
\| \mu - \nu \|_{tv} = \frac{1}{2} \sum_{x \in S} |\mu(x) - \nu(x)| = \sum_{x \in S} (\mu(x) - \nu(x))_+
\]
where \( a_+ = \max(a,0) \). Show that if \( P \) is the transition matrix of an irreducible, aperiodic Markov chain on a state space \( S \) with invariant distribution \( \pi \), and if \( d(t) = \sup_x \| P^t(x, \cdot) - \pi(\cdot) \|_{tv} \) then \( d(t) \leq \bar{d}(t) \leq 2d(t) \) where \( \bar{d}(t) = \sup_x \| P^t(x, \cdot) - P^t(y, \cdot) \|_{tv} \).

(b) Define what is meant by a coupling of \( \mu \) and \( \nu \), and show that if \((X,Y)\) is such a coupling then
\[
\| \mu - \nu \|_{tv} \leq \mathbb{P}(X \neq Y).
\]

(c) Using a coupling or otherwise, show that \( \bar{d}(t + s) \leq \bar{d}(t) \bar{d}(s) \). Hence deduce that \( \rho = \lim_{t \to \infty} d(t)^{1/t} \) exists. [Hint: you can use without proof the following lemma: if \( f \) is subadditive, i.e., if \( f(t + s) \leq f(t) + f(s) \) for all \( s, t \geq 0 \) then \( \lim_{t \to \infty} f(t)/t \) exists in \( \mathbb{R} \cup \{-\infty\} \).]

(d) Assuming also reversibility in the above setting, what is the value of \( \rho \)? [You can use without proof any result from the course, provided it is clearly stated].
Problem 2.

(a) Let $P$ be the transition matrix of an irreducible, aperiodic and reversible Markov chain on a finite state space $S$ of size $n$ with invariant distribution $(\pi(x))_{x \in S}$, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Define the Dirichlet form $\mathcal{E}(f,f)$ associated to $P$, and give without proof an equivalent expression. State and prove the variational characterisation of the spectral gap in terms of $\mathcal{E}(f,f)$. State without proof a similar characterisation for higher order eigenvalues.

(b) Let $P, \tilde{P}$ be two transitive Markov chains on $S$, with corresponding Dirichlet forms $\mathcal{E}, \tilde{\mathcal{E}}$ respectively. Suppose that if $A > 0$ is such that $\tilde{\mathcal{E}}(f,f) \leq A \mathcal{E}(f,f)$. State and prove a theorem concerning their respective mixing behaviours in $L^2$, defining carefully the expressions you introduce. (You can use without proof a relation between eigenvalues and $L^2$ distance to stationarity, provided that this is stated clearly).

(c) Define the interchange process on a connected graph $G = (V,E)$. State a theorem giving a bound of mixing time of the interchange on $G$ in terms of geometric quantities associated with $G$.

(d) Suppose $G = (V,E) = [0,n)^2 \cap \mathbb{Z}^2$ is the $n \times n$ square, and $E$ is the set of nearest neighbour edges, so $(u,v) \in E$ if and only if $\|u-v\|_1 = 1$ for $u, v \in V$ (here $\|u\|_1 = |u_1| + |u_2|$ for $u = (u_1, u_2)$). Show that the interchange process (in continuous time) satisfies $t_{\text{mix}} = O(n^4 \log n)$. On the other hand, explain briefly, e.g. by considering the position of a single card, why $t_{\text{mix}} \geq cn^4$ for some $c > 0$. 

Problem 3.

(a) Let \((X_t, t = 0, 1, \ldots)\) be an irreducible, aperiodic and reversible Markov chain on a finite state space \(S\) with invariant distribution \(\pi(y), y \in S\). Define the notion of \textit{mixing time} \(t_{\text{mix}}(\alpha)\) at level \(\alpha \in (0, 1)\).

Give the definition of the \textit{absolute spectral gap} \(\gamma_*\) of the chain, as well as that of the \textit{relaxation time} \(t_{\text{rel}}\), and give without proof the statement of a relation between relaxation time and mixing time at level \(\varepsilon > 0\).

(b) Define the \textit{bottleneck} (or isoperimetric) \textit{ratio} \(\Phi_*\) of an irreducible, reversible Markov chain on a finite state space \(S\). State Cheeger’s inequality, and prove that if \(\gamma\) is the spectral gap, then \(\gamma \leq 2\Phi_*\).

(c) Let \(S = \{1, \ldots, n\}\) be the \(n\)-cycle and consider the Markov chain on \(S\) which is the lazy simple random walk on \(S\). Show that \(\Phi_* = (1/n)(1 + o(1))\) as \(n \to \infty\). Deduce that \(\gamma \geq (1 + o(1))2/n^2\), and hence show that \(t_{\text{mix}}(1/4) \leq O(n^2 \log n)\).

(d) Compute all the eigenvalues of this Markov chain, and compare the estimate above with the actual value of the spectral gap.
Problem 4.

(a) Let \((X_t, t = 0, 1, \ldots)\) be an irreducible, aperiodic and reversible Markov chain on a finite state space \(S\) with invariant distribution \(\pi(y), y \in S\). Show that if \(P^t(x, y)\) denote the \(t\)-step transition probabilities of the chain,

\[
P^t(x, y) = \frac{\pi(y)}{\pi(x)} = \sum_{j=1}^{n} \lambda_j^t f_j(x) f_j(y)
\]

where \(\lambda_j\) are the eigenvalues and \(f_j\) are functions which you should specify. [You can assume without proof that there exists an orthonormal basis of eigenfunctions for the inner product associated with the \(\ell^2\) norm \(\|f\|_2 = (\sum_x f(x)^2 \pi(x))^{1/2}\).]

(b) Define the relaxation time \(t_{rel}\), and the \(\ell^2\) distance \(d_2(t)\) to equilibrium. Show that \(d(t) \leq (1/2)d_2(t)\) where \(d(t)\) is the total variation distance to equilibrium, and show that

\[
t_{mix}(\varepsilon) \leq \log \left( \frac{1}{2\varepsilon \sqrt{\pi_{min}}} \right) t_{rel}
\]

where \(\pi_{min} = \min\{\pi(x) : x \in S\}\), and \(t_{mix}(\varepsilon)\) is the mixing time at level \(\varepsilon\).

(c) Show that \(P^{2t}(x, x)\) is a decreasing sequence (as a function of \(t = 0, 1, \ldots\)). Show however with an example that \(P^t(x, x)\) is not in general monotone, as a function of \(t\).

(d) Show that \(d_2(t)\) is a contraction: for all \(t, s \geq 0\):

\[
d_2(t+s) \leq d_2(s)e^{-t/t_{rel}}.
\]

[You can use freely without proof the inequality \(1 - x \leq e^{-x}\), valid for all \(x \in \mathbb{R}\).]