Example Sheet 1

1. (a) Let $X$ be an ergodic Markov chain with an invariant measure $\pi$. Define

$$d(t) = \sup_{x \in X} \|P^t(x, \cdot) - \pi(\cdot)\|_{TV}.$$  

Set also $\bar{d}(t) = \sup_{x,y} \|P^t(x, \cdot) - P^t(y, \cdot)\|_{TV}$. Show that

$$d(t) \leq \bar{d}(t) \leq 2d(t).$$

(b) Suppose $X$ is a random walk on a graph of diameter $D$. Show that $t_{\text{mix}} \geq D/2$.

2. Let $X, X'$ be Poisson($\lambda, \lambda'$) respectively, and let $\mu, \mu'$ be their respective law. Express $\|\mu - \mu'\|$ as a sum, and show that this tends to 0 when $\lambda' \to \lambda$.

By a coupling argument or otherwise, show that $\|\mu - \mu'\| \leq |\lambda' - \lambda|$ for all $\lambda, \lambda'$.

3. Consider the coupon collector problem: an urn contains $n$ white balls initially and we sample from the urn with replacement uniformly at random, each time painting the ball black (whatever its colour). Show that the time $\tau_n$ until all balls are black is concentrated near $n \log n$, i.e., $\tau_n/(n \log n) \to 1$ in probability. How many balls are black by time $(1/2)n \log n$?

4. Consider two lazy random walks $(X_t, t = 0, 1, \ldots)$ and $(Y_t, t = 0, 1, \ldots)$ on the $n$-cycle (lazy means the walks stay put with probability $1/2$), started from $x$ and $y$ respectively.

(a) Consider a coupling where $X$ moves with probability $1/2$ or $Y$ moves with probability $1/2$ at each step. Show that the time $\tau$ at which they meet satisfies $\mathbb{E}(\tau) = k(n - k)$, where $k$ is the cyclical distance from $x$ to $y$. [Hint. You can admit without proof the gambler’s ruin estimate from Markov chains: if $Z$ is a simple random walk on $\mathbb{Z}$ started from $k$, then the first hitting time $T$ of $\{0, n\}$ satisfies $\mathbb{E}(T) = k(n - k).$]

(b) Deduce that if $d(t)$ is the total variation distance to stationarity then

$$d(t) \leq \frac{n^2}{2t}.$$  

(c) Hence deduce that $t_{\text{mix}} \leq 2n^2$. Show also that $t_{\text{mix}} \geq cn^2$ for some small constant $c$.

5. (a) Let $X, X'$ be two random variable on a set $S$. Let $\mathbb{P}, \mathbb{Q}$ be their respective law and let $f(x) = d\mathbb{Q}(x)/d\mathbb{P}(x)$ denote the Radon–Nikodym derivative. (If $S$ is discrete, then this is just $f(x) = \mathbb{Q}(x)/\mathbb{P}(x)$. ) Show that

$$\|X - X'\| = \mathbb{E}_X \left| \frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right|$$

(b) Let $\mathbb{P}$ be the law of a random variable $X \sim \text{Binomial}(n, 1/2)$ and let $\mathbb{Q}$ be the law of $Y = (X - m)_+$, where $m = o(n^{1/2})$. By computing the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ explicitly or otherwise, show that

$$\|X - Y\| \to 0$$
as \( n \to \infty \). [If you haven't done measure theory, an intuitive justification will suffice.]

(c) Now consider lazy random walk \((Z_t, t = 0, 1, \ldots)\) on the hypercube \(\{0, 1\}^n\): at each step we select a coordinate uniformly at random, and flip it with probability 1/2. Show that at equilibrium, the number of coordinates equal to 1 is a Binomial \((n, 1/2)\). Let \(\tau\) be the time at which all but \(m\) coordinates have been selected. Show that \(\mathbb{P}(\tau \geq (1/2 + \epsilon)n \log n) \to 0\) as \(n \to \infty\), for any \(\epsilon\).

(d) Show that \(\tau\) is independent from \(Z_\tau\) and that \(\|Z_\tau - \pi\| \to 0\) where \(\pi\) is the equilibrium distribution. \([Hint: what is the number of 1s?]\)

(e) Finally deduce that if \(t = (1/2 + \epsilon)n \log n\), then \(d(t) \to 0\). Deduce that \(Z\) has a cutoff at time \((1/2)n\log n\).

6. (a) Let \(\sigma \in S_n\) be a random uniform permutation. Let \(X\) denote the number of fixed points of \(\sigma\), i.e., the numbers of \(1 \leq i \leq n\) such that \(\sigma(i) = i\). Show that \(\mathbb{E}(X) = 1\) and \(\text{Var}(X) = 1\).

(b) Show that for random transpositions, \(t_{\text{mix}} \geq (1/2 - \epsilon)n \log n\) for any \(\epsilon > 0\) and \(n\) sufficiently large.

7. Random transpositions is the Markov chain such that at each step the left hand picks a card uniformly in the deck, the right hand too, and the two cards are switched. (Note that the two cards can be the same, in which case nothing happens). Formally,

\[
P(\sigma, \sigma') = \begin{cases} 
1/n & \text{if } \sigma = \sigma' \\
2/n^2 & \text{if } \sigma' = \sigma \cdot (i,j) \text{ for some } 1 \leq i \neq j \leq n \\
0 & \text{else}
\end{cases}
\]

Consider the following coupling argument for random transpositions. At each time \(t\), select a label \(X_t\) uniformly at random in \([n] = \{1, \ldots, n\}\) and a position \(Y_t\) uniformly at random in \([n]\), independently. Then at time \(t\), transpose the card with label \(X_t\) together with the card in position \(Y_t\).

(a) Show that this mechanism generates the random transpositions Markov chain.

(b) By coupling to a deck which already starts in equilibrium, show that \(t_{\text{mix}} = O(n^2)\) (consider the set of cards which occupy the same positions in the two decks).

8. (a) Let \(P\) be a transition matrix. Show that if \(\lambda\) is an eigenvalue then \(|\lambda| \leq 1\).

(b) Suppose \(P\) is irreducible, and for \(x \in S\), let \(T(x) = \{t = 0, 1, \ldots : P^t(x, x) > 0\}\). Show that \(T(x) \subset 2\mathbb{Z}\) if and only if \(-1\) is an eigenvalue of \(P\).

9. Show that if \(P\) is a transition matrix of an irreducible aperiodic Markov chain, and if \(\gamma^* = \max\{|\lambda| : \lambda \in \text{Sp}(P), |\lambda| < 1\}\) is the absolute spectral gap, then for any function \(f : S \to \mathbb{R}\),

\[
\text{Var}_\pi(P^t f) \leq (1 - \gamma^*)^{2t} \text{Var}_\pi(f).
\]

10. Let \(T\) be a tree with \(n\) vertices and maximal degree \(d\), diameter \(H\). Show that for some universal constant \(c\), if \(\gamma\) is the spectral gap,

\[
\gamma \geq \frac{c}{d^2 n H}.
\]