Introduction to Random Diffusions

The main reason to study random diffusions is that this class of processes combines two key features of modern probability theory. On the one hand they are semi-martingales so the whole Itô theory of stochastic integration applies, but on the other hand they are also strong Markov processes. The combination of techniques from those two worlds enables us to have an in-depth look at their behavior, particularly in the linear case $d = 1$.

Here we will focus our attention on the notion of infinitesimal generator of a Markov process. For instance, we will see how to compute the exit time of a given domain by solving a P.D.E. that involves the infinitesimal generator. We will present (but not prove !) Dynkin's theorem, which establishes a remarkable connection with the theory of Stochastic Differential Equations. Finally, in the linear case, we will discuss the notion of scale function and speed measure, which we then use to show (in a particular case) that any diffusion is a space-and-time change of a Brownian Motion. This beautiful theorem is the ultimate tool for any given question on the topic !

However several important aspects of the theory will be left aside, such as: S.D.E theory, martingale problems, boundary classification, analytical theory of the semi-group and the generator.

Although we will not be always rigorous here (since we just want to give an overview), the expected background is that of basic probability theory at the level of Itô's formula.

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References:

- Revuz-Yor (1999), chap. VII,
- Rogers-Williams (2000), vol. 1, chap. III, and vol. 2, chap. V.6

1. The infinitesimal generator of a Markov process

• Recall the definition of the semi-group of a Markov process: $P_t f(x) = \mathbf{E}_x[f(X_t)]$. Then $P_{t+s} = P_t P_s$ (semi-group property, which expresses the Markovian nature of X). We will assume that P is a Feller semi-group. This means that P_t sends C^0, the continuous functions vanishing at ∞, to C^0. P_t is then immediately a bounded operator on C^0. The other requirement is that $P_t \to P_0$ as $t \to 0$, in the norm of operators on C^0. But the analytical aspect of Markov processes is not our purpose and so it is not essential that the actual definition be understood for what follows.

• Definition of the generator, and its domain: Let $f \in C_0$. We say that $f \in \mathcal{D}$ if the following limit exists in C_0:

$$Lf := \lim_{\varepsilon \to 0} \frac{P_\varepsilon f - f}{\varepsilon}$$

• Interpretation : The generator describes how the process moves from one point to another in an infinitely small time interval:

$$\mathbf{E}[f(X_{t+h}) - f(X_t)|\mathcal{F}_t] = hLf(X_t) + o(h)$$

• The generator is therefore defined as the derivative of the semi-group at time 0. Thanks to the semi-group property, this extends to the following fact: Let $f \in \mathcal{D}$. Then for each t, $P_t f \in \mathcal{D}$ and moreover $t \mapsto P_t f$ is strongly differentiable in C_0 with derivative $P_t A f$:

$$\frac{d}{dt} P_t = A P_t$$

This is sometimes called the Chapman-Kolmogorov equation.

• Example: The standard Poisson process. It is a Feller process so it has a generator

$$Lf(x) = f(x+1) - f(x)$$

• "Dynkin's formula" : If f is in the domain of the generator:

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$$

is a martingale. This is really the most important fact about generators, it should almost be regarded as a definition. (Actually, this formula is used for the definition of *extended* generator, see later).

• Conversely, if $f \in C_0$ and $\exists g \in C_0$ such that:

$$f(X_t) - f(X_0) - \int_0^t g(X_s)ds$$

is a martingale, then f is in the domain and $Lf = g$.

• Example: Brownian Motion. By Dynkin's formula, and Itô's formula, we conclude that on the space C_0^2, the generator of BM is

$$Lf = \frac{1}{2}\Delta f, f \in C_0^2$$

In fact, when $d = 1$ the domain of the generator is *exactly* the space of C_0^2 functions (see Revuz-Yor, chap. VII, prop. (1.10)). In higher dimensions, this is a larger space.

• Notion of "extended generator", which makes the whole notion of generator very delicate (example of BM and reflected BM). The converse of Dynkin's formula could very well hold for f, g general Borel functions and maybe not in C_0. In this case, you say that f is in the domain of the extended generator and still write $Lf = g$. The point is to be able to do some stochastic calculus and guess what L should look like.

2. Diffusions and Itô processes

• **Definition** There is no consensus in the literature as to what should be called a diffusion process. Here, following Rogers-Williams, a diffusion on an interval I is any continuous Markov process on I with an additional assumption called *regularity* (see below). Following Revuz-Yor, an Itô process is a continuous Markov process whose generator is given by a second-order differential *elliptic* operator:

$$Lf(x) = \frac{1}{2}a(x)\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x), f \in C_K^\infty(I)$$

Theorem 1 *Suppose $I = \mathbf{R}$. A process X is Itô iff it satisfies a S.D.E.*

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

where B is standard Brownian Motion and $\sigma(x)^2 = a(x)$.

• What is behind this theorem ? First, it says that a solution of a SDE is a strong Markov process, a fact that is not so obvious at first. It gives the form of the generator for such a process (it has to be an elliptic 2^d-order differential operator). But, mostly, it says that if X has such a generator, then it must satisfy the above S.D.E. In particular, given any elliptic operator L we can construct a (unique in law) process whose generator is L, provided that the coefficients satisfy some very mild assumptions (like being Lipschitz).

• This theorem is an aspect of the martingale problem formulation of the theory of SDEs, but I will not discuss this subject here. (Basically, it is a way of constructing solutions to SDEs by looking for processes who satisfy Dynkin's formula, i.e. who enjoy many martingale properties).

• A process is called regular if it has a positive probability of hitting any point $y \in I$ starting from any given point $x \in \text{Int}(I)$:

$$\mathbf{P}_x[T_y < \infty] > 0$$

• Notice that Itô processes are regular: therefore they are diffusions. (To see this, use the above theorem in the case $I = \mathbf{R}$, remove the drift by Girsanov's theorem and then apply the Dubins-Schwarz theorem : Brownian Motion visits every point with probability 1. Ellipticity ensures that the Dubins-Schwarz time-change runs through all $\mathbf{R}^+$.

• So Itô processes seem to be very particular diffusions, i.e., diffusions whose generator has a very nice form. However, we have the remarkable theorem due to Dynkin (see Rogers-Williams, III.13.3 for a proof)

Theorem 2 *Let X be any real diffusion on I with generator L. Then the restriction of L to C_K^∞ is a second-order elliptic differential operator:*

$$Lf(x) = \frac{1}{2}a(x)\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x) - c(x)f(x)$$

for some continuous functions $a \geq 0, b$ and $c \geq 0$.

• The case $c(x) \neq 0$ corresponds to killing, or "honesty" of X. If $P_t 1 = 1$, then $c = 0$.

• Interpretation: Dynkin's theorem tells us that, away from the boundary points of I (this is the sense that should be given to the restriction of L to C_K^∞), any continuous and regular Markov process behaves as a solution of a S.D.E. In particular, diffusion and Itô process are the same notion!

• However the word restriction is very important (look at the case of reflected BM, which is a diffusion on $[0, \infty)$ but can NOT satisfy a SDE because of Tanaka's formula).

• Application. Suppose we are interested in computing the exit time of an interval $J \subseteq I$, $\tau_J = \inf\{t > 0 : X_t \notin J\}$, by our diffusion X. Let

$$\phi(x) := \mathbf{E}_x[\tau_J], x \in J$$

Then $\phi(x) < \infty$ and ϕ satisfies the following P.D.E:

$$L\phi(x) = -1, \forall x \in \mathrm{Int}(J)$$

(I have an elementary proof in the case of transient processes or BM - in fact, of processes whose speed measure is infinite, see later. My proof relies on Dynkin's formula and Green's function applied to C^∞ approximations of $\mathbf{1}_{\{J\}}$. For a real proof but more complicated, see Revuz-Yor, Exercise VII.3.17 on Dynkin's operator). The boundary conditions for this P.D.E. depend on J. For instance if $J \subset\subset \mathrm{Int}(I)$, then this is just a Dirichlet problem $\phi(\alpha) = \phi(\beta) = 0$ with $\alpha = \inf J$ and $\beta = \sup J$. Different boundary behavior of the process ("entrance boundary") will induce different boundary conditions for the corresponding P.D.E. affecting for instance the derivative of the function, see the example.

• Example: the 3-dimensional Bessel process $R_t = ||B_t||$ where B is a 3-dimensional Brownian Motion. Say we want to compute the time to reach level 1. By Itô's formula

$$dR_t = dW_t + (1/R_t)dt$$

So the generator of R is simply $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$. If $J = (0, 1)$, and we let $\phi(x) = \mathbf{E}_x[\tau_J]$, then $u = \phi'$ satisfies

$$\frac{1}{2}u' + \frac{1}{x}u = -1$$

Solving the ODE gives $u(x) = (-\frac{2}{3}x^3 - c)x^{-2}$ but we have $u \in C^0$ so $c = 0$ and we get by integration that $\phi(x) = \frac{1}{3}(1 - x^2)$. Therefore, starting from 0, the expected time to reach level 1 is 1/3.

Scale function and Speed measure

This section follows quite closely Rogers-Williams (V.46)

Let X be a regular 1-dimensional diffusion on an interval $I \subseteq \mathbf{R}$.

• **Definition** A scale function $s : I \to \mathbf{R}$ is a strictly increasing function such that for all $a < x < b \in I$,

$$\mathbf{P}_x[T_a > T_b] = \frac{s(x) - s(a)}{s(b) - s(a)}$$

X is said to be in natural scale if $s(x) = x$ is a scale function.

• Scale functions are uniquely determined up to increasing affine functions.

Theorem 3 *Any diffusion X has a scale function s, and $Y = s(X)$ is a diffusion in natural scale on $s(I)$. Besides, if $T = inf\{t > 0 : X_t \notin I\}$, then Y^T is a continuous local $\mathbf{P}_x$-martingale.*

• Conversely, if f is locally bounded such that $f(a) < f(b)$ then f is a scale function of $X^{T_a \wedge T_b}$. (see Revuz-Yor, VII.3.5 but their statement is incorrect as it is).

• Therefore, by Dynkin's formula, a good candidate for a scale function is any nontrivial solution of the P.D.E.

$$Lu(x) = 0, x \in I$$

Note that this agrees well with the fact that scale functions are determined only up to increasing affine transformations.

• Example : BM is in natural scale because it is a martingale. As a result, we get that

$$\mathbf{P}_x[T_a > T_b] = \frac{x-a}{b-a}$$

a fact sometimes known as the *gambler's ruin probability* and that can be obtained directly by stopping B at $T_a \wedge T_b$ and applying the optional sampling theorem.

• From now one we assume that *all diffusions are on natural scale*, which is not too much of a restriction since s is one-to-one.

Theorem 4 *(Itô-McKean, Feller, Dynkin) There exists a Brownian Motion B (on some enrichment of the probability space), and a Radon measure m, non-random, called the speed measure of X, such that*

$$X_t = B(\gamma_t)$$

where γ_t is the right-continuous inverse of the functional

$$A_t = \int_I L_t^x m(dx)$$

and $\{L_t^x, x \in \mathbf{R}, t \geq 0\}$ is a jointly continuous version of the local time process of B.

• Why is this theorem so powerful ? After all, X is in natural scale, so it is almost a martingale, so we already know by the Dubins-Schwarz theorem that it is a time-change of BM.

• The reason has to do with the nature of the time-change. Here γ is a (complicated) functional of the B.M. : if we are given the realization of B we can tell how does γ_t look like. This is certainly not the case in the Dubins-Schwarz theorem where we would need some information on the law of the couple $(B_t, < X >_t)$

to be able to say anything - and this never happens in practice, because B is so abstract and has usually no relation whatsoever with our initial process X. Here the knowledge of the Brownian Motion B is enough to know everything about X.

• Why "speed measure" ? Actually, this is a very bad name, since where m is large X moves rather slowly compared to the Brownian time-scale, and conversely. "If the name of speed measure were not so well established we would be tempted to call it 'sloth' measure !" (Roger-Williams).

• The speed measure satisfies $0 < m(a, b) < \infty$ for any $a, b \in \text{Int}(I)$.

• Conversely, given any such Radon measure, you can construct by the reverse procedure a regular diffusion, which is in natural scale (continuity and regularity are not obvious).

• Speed measure, generator, and SDE coefficients. Suppose once again X is in natural scale. Then we have the following relation:

$$L = \frac{1}{2} \frac{d^2}{dmdx}$$

In case you have trouble reading this equation, suppose that for instance m has a density with respect to dx, say $m(dx) = \rho(x)dx$ and $\rho > 0$ uniformly. Then X satisfies

$$dX_t = \rho(X_t)^{-1/2} dW_t$$

which means that we can read off the speed measure straight from its generator or its SDE !

• Short proof: we have $A_t = \int_I L_t^x m(dx) = \int_0^t \rho(B_s)ds$ by the occupation formula. Therefore, if f is C^2, time-changing the martingale

$$f(B_t) - f(B_0) - \frac{1}{2} \int_0^t f''(B_s)ds$$

gives a martingale

$$f(X_t) - f(X_0) - \frac{1}{2} \int_0^t \rho(X_s)^{-1/2} f''(X_s)ds$$

Therefore, the law of X is a solution to the martingale problem $(\rho^{-1/2}, 0)$ which yields the claim about the SDE solution.

• The speed measure as the equilibrium measure.

Theorem 5 *Suppose m is of finite mass and let $\pi(dx)$ be the normalization of m so that π is a probability measure. Then for each x, as $t \to \infty$*

$$X_t \longrightarrow \pi \textit{ in distribution under } \mathbf{P}_x$$

The argument is a beautiful instance of coupling once you realize that m is invariant with respect to the semi-group of X. (see Roger-Williams V.54.5 for a proof)