Abstract. We study random walks on the giant component of the Erdős-Rényi random graph $\mathcal{G}(n,p)$ where $p = \lambda/n$ for $\lambda > 1$ fixed. The mixing time from a worst starting point was shown by Fountoulakis and Reed, and independently by Benjamini, Kozma and Wormald, to have order $\log^2 n$. We prove that starting from a uniform vertex (equivalently, from a fixed vertex conditioned to belong to the giant) both accelerates mixing to $O(\log n)$ and concentrates it (the cutoff phenomenon occurs): the typical mixing is at $(\nu d)^{-1} \log n \pm (\log n)^{1/2+o(1)}$, where $\nu$ and $d$ are the speed of random walk and dimension of harmonic measure on a Poisson($\lambda$)-Galton-Watson tree. Analogous results are given for graphs with prescribed degree sequences, where cutoff is shown both for the simple and for the non-backtracking random walk.

1. Introduction

The time it takes random walk to approach its stationary distribution on a graph is a gauge for an array of properties of the underlying geometry: it reflects the histogram of distances between vertices, both typical and extremal (radius and diameter); it is affected by local traps (e.g., escaping from a bad starting position) as well as by global bottlenecks (sparse cuts between large sets); and it is closely related to the Cheeger constant and expansion of the graph. In this work we study random walk on the giant component $C_1$ of the classical Erdős-Rényi random graph $\mathcal{G}(n,p)$, and build on recent advances in our understanding of its geometry on one hand, and random walks on trees and on random regular graphs on the other, to provide sharp results on mixing on $C_1$ and on the related model of a random graph with a prescribed degree sequences.

The Erdős-Rényi random graph $\mathcal{G}(n,p)$ is the graph on $n$ vertices where each of the $\binom{n}{2}$ possible edges appears independently with probability $p$. In their celebrated papers from the 1960’s, Erdős and Rényi discovered the so-called “double jump” in the size of $C_1$, the largest component in this graph: taking $p = \lambda/n$ with $\lambda$ fixed, at $\lambda < 1$ it is with high probability (w.h.p.) logarithmic in $n$; at $\lambda = 1$ it is of order $n^{2/3}$ in expectation; and at $\lambda > 1$ it is w.h.p. linear (a “giant component”). Of these facts, the critical behavior was fully established only much later by Bollobás [5] and Luczak [22], and extends to the critical window $p = (1 \pm \epsilon)/n$ for $\epsilon = O(n^{-1/3})$ as discovered in [5].

An important notion for the rate of convergence of a Markov chain to stationarity is its (worst-case) total-variation mixing time: for a transition kernel $P$ on a state-space $\Omega$ with a stationary distribution $\pi$, recall $\|\mu - \nu\|_{TV} = \sup_{A \subset \Omega} [\mu(A) - \nu(A)]$ and write

$$t_{\text{mix}}(\epsilon) = \min \{t : d_{TV}(t) < \epsilon\} \quad \text{where} \quad d_{TV}(t) = \max_{x} \|P^{t}(x, \cdot) - \pi\|_{TV};$$

let $t^{(x)}_{\text{mix}}(\epsilon)$ and $d^{(x)}_{TV}(t)$ be the analogs from a fixed (rather than worst-case) initial state $x$. The effect of the threshold parameter $0 < \epsilon < 1$ is addressed by the cutoff phenomenon, a sharp transition from $d_{TV}(t) \approx 1$ to $d_{TV}(t) \approx 0$, whereby $t_{\text{mix}}(\epsilon) = (1 + o(1)) t_{\text{mix}}(\epsilon')$ for any fixed $0 < \epsilon, \epsilon' < 1$ (making $t_{\text{mix}}(\epsilon)$ asymptotically independent of this $\epsilon$).
In recent years, the understanding of the geometry of $C_1$ and techniques for Markov chain analysis became sufficiently developed to give the typical order of the mixing time of random walk, which transitions from $n$ in the critical window ([25]) to $\log^2 n$ at $p = \frac{\lambda}{n}$ on for $\lambda > 1$ fixed ([3, 14]) through the interpolating order $\varepsilon^{-3} \log^2 (\varepsilon^3 n)$ when $p = \frac{1 + \varepsilon}{n}$ for $n^{-1/3} \ll \varepsilon \ll 1$ ([9]). Of these facts, the lower bound of $\log^2 n$ on the mixing time in the supercritical regime ($\lambda > 1$) is easy to see, as $C_1$ w.h.p. contains a path of some $c \log n$ degree-2 vertices (escaping this path when started at its midpoint would require $(\varepsilon^3 \log n)^2$ steps in expectation). The two works that independently obtained a matching $\log^2 n$ upper bound used very different approaches: Fountoulakis and Reed [14] relied on a powerful estimate from [13] on mixing in terms of the conductance profile (following Lovász-Kannan [20]), while Benjamini, Kozma and Wormald [3] used a novel decomposition theorem of $C_1$ as a “decorated expander.”

Considering that bottlenecks such as the aforementioned long path are rare in $C_1$, and that their effect is very much tied to the initial vertex of the random walk (starting at the endpoint of said path should already avoid its bottleneck), one may expect accelerated mixing from almost every initial vertex $v_1$. This is indeed the case, and moreover $t_{\text{mix}}^{(v_1)}(\varepsilon)$ then concentrates on $c_0 \log n$ for $c_0$ independent of $\varepsilon$ (cutoff occurs):

**Theorem 1.** Let $C_1$ denote the giant component of the random graph $G(n, p = \lambda/n)$ for $\lambda > 1$ fixed, and let $\nu$ and $d$ denote the speed of random walk and the dimension of harmonic measure on a Poisson$(\lambda)$-Galton-Watson tree, resp. For any $0 < \varepsilon < 1$ fixed, w.h.p. the random walk from a uniformly chosen vertex $v_1 \in C_1$ satisfies

$$|t_{\text{mix}}^{(v_1)}(\varepsilon) - (\nu d)^{-1} \log n| \leq \log^{1/2+o(1)} n.$$  

(1.1)

In particular, w.h.p. the random walk from $v_1$ has cutoff with a $\log^{1/2+o(1)} n$ window.
Before we examine the roles of \( \nu \) and \( d \) in this result, it is helpful to place it in the context of the structure theorem for \( C_1 \), recently given in [9] (see Theorem 3.2 below): a contiguous model for \( C_1 \) is given by (i) choosing a kernel uniformly over graphs on degrees i.i.d. Poisson truncated to be at least 3; (ii) subdividing every edges via i.i.d. geometric variables; and (iii) hanging i.i.d. Poisson Galton-Watson (GW) trees on every vertex. Observe that Steps (ii) and (iii) introduce i.i.d. delays with an exponential tail for the random walk; thus, starting the walk from a uniform vertex (rather than on a long path or a tall tree) would essentially eliminate all but the typical \( O(1) \)-delays, and it should rapidly mix on the kernel (w.h.p. an expander) in time \( O(\log n) \); see Fig. 1.

It is well-known (see [6,11,17,28]) that \( \bar{D} \), the average distance between two vertices in \( C_1 \), is \( \log_3 n + O_p(1) \), analogous to the fact that \( \bar{D} = \log_{d-1} n + O_p(1) \) in \( G(n,d) \), the uniform \( d \)-regular graph on \( n \) vertices (both are locally-tree-like: \( G(n,d) \) resembles a \( d \)-regular tree while \( G(n,p) \) resembles a Poisson(\( \lambda \))-GW tree). It is then natural to expect that \( t_{\text{mix}}^{(v_1)} \) coincides with the time it takes the walk to reach this typical distance from its origin \( v_1 \), which would be \( \nu^{-1} \bar{D} \) for a random walk on a Poisson(\( \lambda \))-GW tree.

Supporting evidence for this on the random 3-regular graph \( G(n,3) \) was given in [4], where it was shown that the distance of the walk from \( v_1 \) after \( t = c \log n \) steps is w.h.p. \( (1 + o(1))(\nu t \wedge \bar{D}) \), with \( \nu = \frac{1}{3} \) being the speed of random walk on a binary tree. Durrett [11, §6] conjectured that reaching the correct distance from \( v_1 \) indicates mixing, namely that \( t_{\text{mix}}^{(v_1)}(\frac{1}{2}) \sim 2\nu^{-1} \bar{D} \) for the lazy (hence the extra factor of 2) random walk on \( G(n,3) \). This was confirmed in [21], and indeed on \( G(n,d) \) the simple random walk has \( t_{\text{mix}}(\epsilon) = \frac{d}{\nu} \log_{d-1} n + O(\sqrt{\log n}) \), i.e., there is cutoff at \( \nu^{-1} \bar{D} \) with an \( O(\sqrt{\log n}) \) window (in particular, random walk has cutoff on almost every \( d \)-regular; prior to the work [21] cutoff was confirmed almost exclusively on graphs with unbounded degree).

However, Theorem 1 shows that \( t_{\text{mix}} \sim (\nu d)^{-1} \log n \) vs. the \( \nu^{-1} \log_3 n \) steps needed for the distance from \( v_1 \) to reach its typical value (and stabilize there; see Corollary 3.3). As it turns out, the “dimension drop” of harmonic measure (whereby \( d < \log \mathbb{E}Z \) unless

\footnote{Here \( O_p(1) \) denotes a random variable that is bounded in probability.}
Figure 3. Harmonic measure on the first 7 generations of a GW-tree with offspring distribution $p_1 = p_3 = \frac{1}{2}$, as apparent on 16 generations.

The offspring distribution $Z$ is a constant), discovered in [23], plays a crucial role here, and stands behind this slowdown factor of $\left(\frac{d}{\log \lambda}\right)^{-1} > 1$. Indeed, while generation $k$ of the GW-tree has size about $\left\langle E \right\rangle^k$, random walk at distance $k$ from the root concentrates on an exponentially small subset of size about $\exp(dk)$ (see Fig. 2 and 3).

Hence, $(\nu d)^{-1} \log n$ is certainly a lower bound on $t_{\text{mix}}$ (the factor $\nu^{-1}$ translates time to the distance from the root), and Theorem 1 shows this bound is tight on $G(n, p)$.

The same phenomenon occurs in general for random walk on a graph uniformly chosen out of those with prescribed locally-finite degrees under some mild conditions (such as a Poisson degree distribution, or fixed degrees and proportions (e.g., half the degrees 3 and half 4); Fig. 3 matches degrees 2 and 4 with proportions $\frac{2}{3}$ and $\frac{1}{3}$, resp.), whereby starting from a fixed initial vertex again induces cutoff around $(\nu d)^{-1} \log n$.

**Theorem 2.** Let $G$ be a random graph on a degree sequence $(d_1, \ldots, d_n)$, denote its frequencies by $p_k = \frac{1}{n} |\{i : d_i = k\}|$ for $k \geq 1$, and define $Z$ by $P(Z = k - 1) \propto kp_k$. Set $t_* = (\nu d)^{-1} \log n$, where $\nu$ and $d$ are the speed of random walk and dimension of harmonic measure on a Galton-Watson tree with offspring distribution $Z$.

(i) Set $w_n = \sqrt{\log n}$ and suppose that for some absolute constants $K$ and $\delta > 0$,

$$\mathbb{E}Z < K, \quad 2 \leq Z \leq \Delta = \exp \left(\left(\log n\right)^{1/2-\delta}\right).$$

For any $\varepsilon > 0$ there exists some $\gamma > 0$ such that, if $n$ is large enough, w.h.p.

$$d_{TV}^{(w_n)}(t_* - \gamma w_n) > 1 - \varepsilon,$$

whereas

$$d_{TV}^{(w_n)}(t_* + \gamma w_n) < \varepsilon.$$

(ii) Under the weaker assumption that for some absolute constants $K$ and $\delta > 0$,

$$p_2 < 1 - \delta, \quad 1 + \delta < \mathbb{E}Z < K, \quad Z \leq \Delta = \exp \left(\left(\log n\right)^{1/2-\delta}\right),$$

we have that, w.h.p. on the event that $v_1$ belongs to the largest component of $G$, the statement in (1.3) holds for $w_n = \sqrt{\log n (\log \log n)^2}$.
The intuition behind these results is better seen for the non-backtracking (as opposed to simple) random walk (NBRW), which, upon arriving to a vertex $v$ from some other vertex $u$, moves to a uniformly chosen neighbor $w \neq u$ (formally, this is a Markov chain whose state-space is the set of directed edges in the graph). This walk has speed $\nu = 1$ on a GW-tree (as it never backtracks towards the root), and on $G(n, 3)$ it was shown in [21] to satisfy $|t_{\text{mix}}(\epsilon) - \log_3 n| < C(\epsilon)$ for any fixed $0 < \epsilon < 1$ w.h.p.—indeed, cutoff (with an $O(1)$-window) occurs once the distance from the origin reaches the average graph distance. If we instead take a random graph on $2n/3$ vertices of degree 2 and $n/3$ vertices of degree 4, this corresponds to a GW-tree with an offspring distribution $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 3) = \frac{1}{4}$; since its $k$-th generation grows as $2^k$, the distance between two typical vertices is again asymptotically $\log_3 n$. However, the probability that the NBRW follows a given path $v_1, v_2, \ldots, v_k$ is $\prod_i 1/Z_i$ (with $Z_i$ denoting the number of children of $v_i$); setting $d = \mathbb{E} \log Z = \log \sqrt{3} < \log 2$, observe that $\sum_{i<k} \log Z_i$ concentrates around $kd$ by CLT, and hence $k \sim d^{-1} \log n$ is a lower bound on mixing.

(More generally, by Jensen’s inequality $d = \mathbb{E} \log Z < \log \mathbb{E} Z$ unless $Z$ is constant.)

This straightforward description of harmonic measure for the NBRW allows one to directly control the location of this walk in the random graph. Consequently, by adding a few ingredients to the approach originally used in [21], we were able to extend the NBRW analysis of that work to the non-regular setting (Theorems 4.1–4.2; similar results for the NBRW were independently obtained in [2]; see §4 for further details).

However, harmonic measure for simple random walk (SRW) remains mysterious, e.g., there is no explicit formula for $d$ even for fairly simple offspring distributions $Z$. Formally, let $T$ be an infinite GW-tree rooted at $\rho$ with offspring distribution $Z$. Under our assumptions the random walk $(X_t)$ is transient, so its loop-erasure defines a unique ray $\xi$. Define also its asymptotic speed $\nu$ as follows (well-defined for almost every tree)

$$\nu \overset{a.s.}{=} \lim_{t \to \infty} \frac{1}{t} \text{dist}(X_t, \rho),$$

and define the Hausdorff dimension of harmonic measure to be

$$d \overset{a.s.}{=} \lim_{t \to \infty} \frac{1}{t} W_T(\xi_t)$$

where

$$W_T(v) = -\log \mathbb{P}(v \in \xi).$$

(While written here as a Hölder exponent, one may interpret it as a Hausdorff dimension by endowing the set of all rays $\partial T$ with the metric $d(\xi, \eta) = \exp(-|\xi \wedge \eta|)$, in which $\xi \wedge \eta$ denotes the longest common prefix of the two rays $\xi$ and $\eta$; see [27, §14 and §16].)

With this notation, it was shown in [23] that in the probability space $\text{GW} \times \text{SRW},$

$$W_T(\xi_t)/t \overset{a.s.}{\to} d.$$

Building on the results of [23] as well as [7, 24], we establish the following refinement of (1.6), which plays a central role in our proof and seems to be of independent interest.

**Proposition 3.** Let $T$ be a Galton-Watson tree, conditioned to survive, whose offspring distribution $Z$ satisfies $1 < \mathbb{E} Z < \infty$. Then

$$\mathbb{E}[W_T(\xi_t)] = dt + O(1) \quad \text{and} \quad \text{Var}(W_T(\xi_t)) = O(t).$$

Indeed, the upper bound on $t_{\text{mix}}$ will hinge on showing that w.h.p. at a suitable time $t = ((\nu d)^{-1} \log n) + O(\sqrt{\log n})$ the $L^2$-distance of SRW from equilibrium is $\exp(O(\sqrt{\log n}))$—a term that originates from the $O(\sqrt{t})$-fluctuations of $W_T(\xi_t)$ as per Proposition 3.
Organization and notation. In §2 we will establish Proposition 3 along with several other estimates for random walk on GW-trees, building on the works of [7, 23, 24]. Section 3 studies SRW on random graphs and contains the proofs of Theorem 1 and 2, while Section 4 is devoted to the analysis of the NBRW.

Throughout the paper, a sequence of events $A_n$ is said to hold with high probability (w.h.p.) if $\mathbb{P}(A_n) \to 1$ as $n \to \infty$. We use the notation $f = o(g)$ and $f \lesssim g$ to abbreviate $f = O(g)$ and $f = O(g)$, resp. (as well as their converse forms), and $f \gtrsim g$ to denote $f \gtrsim g \gtrsim f$. Finally, in the context of an offspring distribution $Z$ with $\mathbb{E} Z > 1$, we say that $T$ is a GW$^*$-tree to refer to the corresponding GW-tree conditioned on survival.

2. Random walk estimates on Galton-Watson trees

Let $T$ be a fixed infinite tree $T$ rooted at some vertex $\rho$ on which random walk is transient. In what follows, we will always use $(X_t)$ to denote SRW on $T$ and let the random variable $\xi \in \partial T$ denote its limit, i.e., the unique ray that $(X_t)$ visits i.o., or equivalently, the loop-erased trace of $(X_t)_{t=0}^\infty$. For any vertex $v \in T$ other than the root, we let $v^-$ denote its parent in $T$, and let $\theta_T(v)$ denote the incoming flow at $v$ relative to its parent corresponding to harmonic measure:

$$\theta_T(v) = \mathbb{P}(v \in \xi \mid v^- \in \xi), \quad (2.1)$$

so that if $(v_0 = \rho, v_1, \ldots, v_k = v)$ is a shortest path in $T$ then $\mathbb{P}(v \in \xi) = \prod_{i=1}^k \theta_T(v_i)$.

Our goal in this section is to establish Proposition 3 as well as the next two estimates, in each of which the underlying offspring distribution $Z$ is assumed to have $\mathbb{E} Z > 1$.

Lemma 2.1. Let $T$ be a GW$^*$-tree. There exists $c > 0$ so that for any $R > 0$ the following holds with probability $1 - O(\exp(-cR))$. If $v \in T$ and $\mathcal{F}$ are such that $T_R(v^-)$, the depth-$R$ subtree of $T$ rooted at the parent of $v$, is $\mathcal{F}$-measurable, then

$$\left| \theta_T(v) - \mathbb{E}[\theta_T(v) \mid \mathcal{F}] \right| \leq \exp(-cR).$$

Lemma 2.2. Let $T$ be a GW$^*$-tree. There exists $c > 0$ such that, for any $R, t > 0$,

$$\mathbb{P}(\text{dist}(X_t, \xi) > R) \leq \exp(-cR).$$

2.1. Proof of Lemma 2.1. Recalling that $\theta_T(v)$ is the conditional probability that $v \in \xi$ given that $v^- \in \xi$, we may assume without loss of generality that $\rho = v^-$, whence

$$\theta_T(v) = \mathbb{P}(\xi_1 = v) .$$

Our first step will be to approximate the relative flow $\theta_T(v)$ by harmonic measure at depth-$(R - 1)$, that is, by the probability that the first visit of SRW to the leaves of $T_R(\rho)$ is in a subtree of $v$. Formally, denoting the children of $\rho$ as $u_1, u_2, \ldots, u_r$ where $u_1 = v$, let $L_i$ be the set of leaves of the depth-$(R - 1)$ subtree of $u_i$ (i.e., descendants of $u_i$ at distance $(R - 1)$ from it) and let $\tau = \min\{t : X_t \in \cup L_i\}$ be the hitting time of SRW to either of these sets. With this notation, define

$$\hat{\theta}_T(v) = \mathbb{P}(X_\tau \in L_1).$$
Now let $B = \bigcup_{k>0} \{ X_{\tau+k} = \rho \}$ denote the event that $X_\tau$ revisits $\rho = v^-$ after time $\tau$. Clearly, on the event $B^c$, we have $\xi_1 = v$ iff $X_\tau \in L_1$, and so

$$\left| \theta_T(v) - \tilde{\theta}_T(v) \right| \leq \mathbb{P}(B).$$

Estimating $\mathbb{P}(B)$ follows from the following result in [7, p21], which was obtained as a corollary of a powerful lemma of Grimmett and Kesten [16] (cf. [7, Lemma 2.2]).

**Lemma 2.3.** There exist $c, \alpha > 0$ such that the following holds. Let $T$ be a GW*-tree and let $R \geq 1$. With probability at least $1 - \exp(-cR)$, every depth-$R$ vertex $x$ satisfies that the ray $P$ from the root to $x$ has at least $\alpha R$ vertices $v$ such that $\mathbb{P}_v(\tau_P^+ = \infty) > c$, where $\tau_P^+$ denotes the return time of RW to the ray $P$.

By this lemma, with probability $1 - O(\exp(-cR))$ the tree $T$ is such that every $z \in \cup L_i$ satisfies the following: there are at least $\alpha R$ vertices on its path to $\rho$ so that SRW from such a vertex $y$ has a probability $q > 0$ of escaping to $\infty$ never again visiting its parent $y^-$. In particular, $\mathbb{P}(B) \leq (1 - q)^{\alpha R} \leq \exp(-c'R)$ for some $\epsilon' > 0$.

Finally, note that $\tilde{\theta}_T(v)$ is $\mathcal{F}$-measurable by our hypothesis that the depth-$R$ subtrees of $u_1, \ldots, u_r$ are all $\mathcal{F}$-measurable; thus, Jensen’s inequality concludes the proof:

$$\mathbb{E} \left[ \theta_T(v) \mid \mathcal{F} \right] - \mathbb{E} \left[ \tilde{\theta}_T(v) \mid \mathcal{F} \right] \leq \mathbb{E} \left[ |\theta_T(v) - \tilde{\theta}_T(v)| \right] \leq e^{-c'R}.$$

### 2.2. Proof of Lemma 2.2.

Following [23], consider the Augmented GW-tree (AGW), which is obtained by joining the roots of two i.i.d. GW-trees by an edge. A highly useful observation of [23] is that the process in which SRW acts on $(T, \rho)$ by moving the root to one of its neighbors is stationary when started at an AGW-tree $T$ and one of its roots. Clearly, the probability of the event under consideration here in the GW-tree is, up to constant factors from below and above, the same as the analogous probability in the AGW-tree (with positive probability the walk never traverses the edge joining the two copies; conversely, a union bound can be applied to the two GW-tree instances).

For brevity, an AGW*-tree will denote the AGW corresponding to two GW*-trees.

This stationarity reduces our goal to showing that $\mathbb{P}(\text{dist}(X_0, \xi) > R) \leq \exp(-cR)$, with $\xi$ now a bi-infinite path, and the walk having both a past and a future. Expose the past of the SRW walk, let $\xi'$ denote its loop-erasure, and let $v'$ denote the vertex of $\xi'$ at distance $R$ from the root. On the event $\text{dist}(X_0, \xi) > R$, the future of the walk must revisit $v'$ (so as to loop-erase it from $\xi$). However, by Lemma 2.3, with probability $1 - O(\exp(-cR))$ the AGW*-tree $T$ satisfies that the path from $\rho$ to $v'$ contains some $\alpha R$ vertices from which the walk would escape and never return to this path with probability bounded away from 0, whence the probability of revisiting $v'$ is at most $\exp(-cR)$ for some $c > 0$.

### 2.3. Proof of Proposition 3.

A regeneration point for the SRW on the GW*-tree is a time $\tau \geq 1$ such that the edge $(X_{\tau-1}, X_\tau)$ is traversed exactly once along the trace of the random walk. Set $\tau_0 = 0$ and let $\tau_1 < \tau_2 < \ldots$ denote the sequence of regeneration points on a random GW*-tree $T$. The following remarkable result due to Kesten was reproduced in [26] (see also [23,24] where the first part was observed).
Lemma 2.4. Let $T$ be a GW*-tree rooted at $\rho$, let $(\tau_i)_{i \geq 0}$ be the regeneration points of SRW on it, and let $\varphi_i = \text{dist}(\rho, X_{\tau_i})$ denote their depths in the tree.

(a) Let $T_i$ ($i \geq 1$) denote the depth-$\varphi_i$ tree rooted at $X_{\tau_{i-1}}$. Then \(\{T_i, (X_t)_{t \leq \tau_i}\}_{i \geq 1}\) are mutually independent, and furthermore for $i \geq 2$ they are i.i.d.

(b) There exist some $\alpha > 0$ such that $\mathbb{E}\exp[\alpha(\varphi_i - \varphi_{i-1})] < \infty$.

Let $\varphi_i = \text{dist}(\rho, X_{\tau_i})$ ($i \geq 1$) as in the above lemma (noting that necessarily $X_{\tau_i} = \xi_{\varphi_i}$ as regeneration points are by definition part of the loop erased trace $\xi$) and set

\[ Y_i = -\log \theta_T(\xi_i) \quad \text{and} \quad B_i = \sum_{\varphi_{i-1} < k \leq \varphi_i \wedge t} Y_k, \]

so that our goal is a CLT for $\sum_{i=1}^t Y_i = W_T(\xi_t)$. Observe that $Y_i$ has exponential moments, since, e.g., if $u_1, \ldots, u_{Z_1}$ are the children of the root then for any $0 \leq r \leq 1$,

\[ \mathbb{E}\exp(r Y_1) = \mathbb{E}[Z_1] \mathbb{E}[\theta_T(u_1) \exp[r \log(1/\theta_T(u_1))]] \leq \mathbb{E}Z = O(1). \]

Consequently, $B_i$ has exponential moments thanks to the exponential tail of $\varphi_i - \varphi_{i-1}$ (see Lemma 2.4); in particular, $\mathbb{E}B_1 = O(1)$, and since one can couple the distributions AGW*-SRW and GW*-SRW once the walk exits the first regeneration block (beyond which both are GW*-trees conditioned on SRW never traversing the regeneration edge), it will suffice to bound the variance when $T$ is an AGW*-tree. Note that in that setting $\mathbb{E}[W_T(\xi_t)] = dt$, as was shown in [23] (thus $\mathbb{E}[W_T(\xi_t)] = dt + O(1)$ on the GW*-tree).

Now, since the sequence $(Y_i)_{i \geq 1}$ becomes stationary under AGW*-SRW, it will suffice to show there exist some constants $c, c' > 0$ such that $\text{Cov}(Y_1, Y_k) \leq c \exp(-c'k)$ for all $k$. Indeed, let $T_{[k/2]}$ be the depth-$[k/2]$ subtree of $T$ (sharing the same root), and write $\theta(v) = \theta_T(v)$ and $\theta'(v) = \theta_{T_{[k/2]}}(v)$ for brevity. Denoting the children of the root by $u_1, \ldots, u_{Z_1}$, recall from Lemmas 2.1 and 2.4 that there exists some $c_1, c_2 > 0$ (depending only on the offspring distribution $Z$) such that the events $\mathcal{E}_1, \mathcal{E}_2$ given by

\[ \mathcal{E}_1 = \bigcup_{1 \leq i \leq Z_1} \left\{ |\theta'(u_i) - \theta(u_i)| > e^{-c_1 k} \right\}, \quad \mathcal{E}_2 = \{ \varphi_1 \geq k/2 \} \]

satisfy $\mathbb{P}(\mathcal{E}_1 \cup \mathcal{E}_2) \leq \exp(-c_2 k)$. It follows from Hölder’s inequality that

\[ \text{Cov}(Y_1 \mathbf{1}_{\mathcal{E}_1}, Y_k) \leq \mathbb{E}[Y_1^2]^{1/2} \left[ \mathbb{E}[Y_k^4] \right]^{1/2} \mathbb{P}(\mathcal{E}_1)^{1/2} \lesssim \exp(-c_2 k) \]

(using that $Y_1$ has finite moments of every order), and similarly for $\text{Cov}(Y_1, Y_k \mathbf{1}_{\mathcal{E}_2})$; hence, it remains to bound $\text{Cov}(Y_1 \mathbf{1}_{\mathcal{E}_1}, Y_k \mathbf{1}_{\mathcal{E}_2})$.

On the event $\mathcal{E}_1$, either (i) we have $\theta(u_i) \leq \exp[-\frac{3}{4} c_1 k]$, whence, e.g.,

\[ \left( \frac{1}{\theta(u_i)} - \frac{1}{\theta'(u_i)} \right) \lesssim e^{-\frac{1}{2} c_1 k} \quad \text{and} \quad \theta'(u_i) \log^2 \frac{1}{\theta(u_i)} \lesssim e^{-\frac{1}{2} c_1 k}, \]

or (ii) we have $\theta(u_i) \geq \exp[-\frac{3}{4} c_1 k]$ and infer from the bound on $|\theta'(u_i) - \theta(u_i)|$ that

\[ \left| \log \frac{1}{\theta(u_i)} - \log \frac{1}{\theta'(u_i)} \right|^2 \leq \left( \frac{e^{-c_1 k}}{\theta(u_i) \wedge \theta'(u_i)} \right)^2 \lesssim e^{-\frac{1}{2} c_1 k}, \]
where in both cases the implicit constant are independent of $k$. Hence, if we let $Y'_1$ assume the value $\log(1/\theta'(u_i))$ with probability $\theta'(u_i)$ for each child $u_i$ of the root of $T$,

$$\mathbb{E}|Y_1 \mathbf{1}_{\xi_1} - Y'_1|^2 \lesssim e^{-\frac{1}{2}c_1 k} \mathbb{E}Z \lesssim e^{-\frac{1}{2}c_1 k};$$

thus (recalling that $\mathbb{E}Y_k^2 = O(1))$,

$$\text{Cov}(Y_1 \mathbf{1}_{\xi_1} - Y'_1, Y_k \mathbf{1}_{\xi_2}) \lesssim \sqrt{\mathbb{E}(Y_1 \mathbf{1}_{\xi_1} - Y'_1)^2 \mathbb{E}Y_k^2 \lesssim e^{-\frac{1}{2}c_1 k} \cdot \mathbb{E}Z}.$$

Thanks to the discrete renewal theorem—crucially using that the renewal intervals $\varphi_i$ have exponential moments (see [18] as well as [19, §IL4])—we can couple $Y_k \mathbf{1}_{\xi_2}$ with an i.i.d. copy of $Y_1$ with probability $1 - O(\exp(-c_3 k))$, and so $\text{Cov}(Y'_1, Y_k \mathbf{1}_{\xi_2}) \lesssim \exp(-c_3 k)$. This completes the desired bound on $\text{Cov}(Y_1, Y_k)$ and concludes the proof. ■

3. Random walk from a typical vertex in a random graph

3.1. Random graph with a prescribed degree sequence. We begin by studying the SRW and LERW on an infinite GW-tree along what would be the cutoff window. This analysis will then carry over to SRW on $G$ via coupling arguments, and provide initial control over the distribution of the walk within the cutoff window. The final step will be to boost mixing using extra time in that window via the graph spectrum.

3.2. Quantitative estimates on the infinite tree. Fix $\varepsilon > 0$ and set

$$m = n \exp \left(-\gamma \sqrt{\log n} \right), \quad \text{(3.1)}$$

$$\ell_0 = \frac{1}{d} \log m = \frac{1}{d} \log n - O(\sqrt{\log n}), \quad \text{(3.2)}$$

$$\ell_1 = \ell_0 + \frac{1}{4d} \gamma \sqrt{\log n}, \quad \text{(3.3)}$$

where $\gamma > 0$ is a suitably large constant with the following two properties:

(i) Recall from Proposition 3 that $|\mathbb{E}W_T(\xi_t) - dt| < C_\varepsilon$ and $\text{Var}(W_T(\xi_t)) \leq C_\varepsilon t$ for some fixed $C_\varepsilon > 0$ depending only on the distribution of $Z$. As $\ell_0, \ell_1 \gg \log n$, by choosing $\gamma$ large enough (in terms of $C_\varepsilon$ and $\varepsilon$), Chebyshev’s inequality yields

$$\mathbb{P}\left(W_T(\xi_{\ell_0}) > \log m + \frac{1}{3} \gamma \sqrt{\log n} \right) > 1 - \varepsilon, \quad \text{(3.4)}$$

$$\mathbb{P}\left(W_T(\xi_{\ell_1}) < \log m + \frac{1}{3} \gamma \sqrt{\log n} \right) > 1 - \varepsilon. \quad \text{(3.5)}$$

(ii) It is known (see [26]) that the distance of SRW $(X_t)$ from the root (its origin) of GW-a.e. tree has mean $\nu t + O(1)$ and variance at most $C'_\varepsilon t$ for some other fixed $C'_\varepsilon > 0$ (which again depends only on the offspring distribution $Z$). Thus,

$$\mathbb{P}\left(\text{dist}(\rho, X_t) - \nu t \leq \frac{1}{16 \varepsilon \gamma} \sqrt{\log n} \right) > 1 - \varepsilon \quad \text{for any } t \leq \nu^{-1} \ell_1$$

provided $\gamma$ is large enough in terms of $d, C'_\varepsilon$ and $\varepsilon$. Combined with the exponential tails of the regeneration times (recall §2.3), taking $\gamma$ large enough further gives

$$\mathbb{P}\left(\max_{t \leq \nu^{-1} \ell_1} |\text{dist}(\rho, X_t) - \nu t| \leq \frac{1}{16 \varepsilon \gamma} \sqrt{\log n} \right) > 1 - \varepsilon. \quad \text{(3.6)}$$
Let $c_0 > 0$ be some large enough constant, and set
\[ R = c_0 \log \log n. \]  

Consider the exploration process where at step $i$ — corresponding to $F_i$ in the filtration — we expose the depth-$i$ descendants of the root, as in the usual breadth-first-search, with one modification: we explore the children of a vertex $v$ only if its distance-$R$ ancestor, denoted $u$, satisfies
\[ \mathbb{E}\left[\mathbb{P}(u \in \xi) \mid F_i \right] \geq \frac{1}{m} \exp\left(-\frac{1}{2} \gamma \sqrt{\log n}\right). \]  

Instead, if (3.8) is violated, we declare that the exploration process is **truncated** at all the depth-$R$ descendants of $u$, and denote this $F_i$-measurable event by $B_{tr}(u)$.

**Lemma 3.1.** Let $T$ be a GW$^*$-tree, and define $\ell_1$, $\gamma$ and $B_{tr}$ as above. Then
\[ \mathbb{P}\left(\bigcup_{t \leq \ell_1} B_{tr}(\xi_t)\right) \leq \mathbb{P}\left(W_T(\xi_{\ell_1}) > \log m + \frac{1}{3} \gamma \sqrt{\log n}\right) < \varepsilon. \]

**Proof.** Suppose $\xi_t = v$ at stage $i$ of the exploration process, during which we consider whether to truncate the process at $v$. Let $u = \xi_{t-R}$ be the distance-$R$ ancestor of $u$. By definition, the fact that we have already reached the vertex $v$ (without truncation along the path leading to it) implies that $T_R(u)$ is $F_i$-measurable, as is the case for every ancestor of $u$, and so if $\rho = v_0, v_1, \ldots, v_k = u$ is the ray from the root to $u$ then
\[ \left| \mathbb{E}\left[\theta_T(v_j) \mid F_i\right] - \theta_T(v_j) \right| \leq \exp(-cR) \]  
for all $j$ thanks to Lemma 2.1. In particular,
\[ \left| \log \mathbb{E}\left[\mathbb{P}(u \in \xi) \mid F_i\right] - W_T(u) \right| \leq \exp(-cR) \sum_{j=1}^{k} \frac{1}{\theta_T(v_j)}. \]

For a given $j$ we know (an immediate corollary of [23, Lemma 9.1]) that
\[ \mathbb{E}[1/\theta_T(v_j)] \leq \frac{\mathbb{E}Z}{1 - \mathbb{E}[1/Z]} = O(1), \]
and so, for instance, $\sum_{j \leq k} 1/\theta_T(v_j) \leq k \log^2 n$ with probability $1 - O(k/\log^2 n)$. Taking $k \leq \ell_1 = O(\log n)$ we find that w.h.p.
\[ \left| \log \mathbb{E}\left[\mathbb{P}(u \in \xi) \mid F_i\right] - W_T(u) \right| \lesssim \exp(-cR) \log^3 n < \frac{1}{\log n} \]  

provided the constant $c_0$ in the definition (3.7) is suitably large (namely, at least $4/c$). Together with the criterion (3.8) this shows that $B_{tr}(u)$ occurs only if
\[ W_T(u) \geq \log m + \frac{1}{2} \gamma \sqrt{\log n} - \frac{1}{\log n} > \log m + \frac{1}{3} \gamma \sqrt{\log n} \]  
for large $n$. The desired inequality now follows from the fact that $W_T(\xi_t)$ is increasing in $t$ (thus it suffices to consider this condition for $W_T(\xi_{\ell_1})$ combined with (3.5).}
Consider the exploration process after some \( k \leq \ell_1 \) rounds, and look at the set of vertices \( S_k \) at distance \( k \) from the root, marking \( S'_k \subset S_k \) as the subset of its vertices for which \( \mathcal{B}_{tr} \) does not occur (a vertex \( v \) may belong to \( S_k \setminus S'_k \), whence the exploration of its subtree will be truncated within at most \( R \) additional levels). The above lemma says that \( \mathbb{P}(\xi_k \in S'_{k}) \geq 1 - \varepsilon \). As argued in the proof of that lemma, we see from the criterion (3.8) and the estimate in (3.9) that if \( \mathcal{B}_{tr}(u) \) does not occur (i.e., the inequality (3.8) is satisfied) then

\[
W_T(u) < \log m + \frac{1}{2} \gamma \sqrt{\log n} + \frac{1}{\log n} \log m + \frac{2}{3} \gamma \sqrt{\log n}.
\]

By the definition of \( W_T(u) = -\log \mathbb{P}(u \in \xi) \), this means that every \( u \in S'_k \) satisfies

\[
\mathbb{P}(u \in \xi) > \frac{1}{m} \exp \left( -\frac{2}{3} \gamma \sqrt{\log n} \right),
\]

and as the events \( \{ u \in \xi \} \) are pairwise disjoint for \( u \in S'_k \), it follows from (3.1) that

\[
|S'_k| \leq m \exp \left( \frac{2}{3} \gamma \sqrt{\log n} \right) = n \exp \left( -\frac{2}{3} \gamma \sqrt{\log n} \right).
\]

In particular, as the maximum degree is at most \( \Delta = \exp[(\log n)^{1/2 - \delta}] \), the number of vertices reached in the first \( \ell_1 \) exploration rounds (i.e., those revealed in \( F_{\ell_1} \)) is at most

\[
\sum_{i \leq \ell_1} |S_i| \leq \sum_{i \leq \ell_1} |S'_i| \Delta^R \leq \ell_1 n \exp \left( -\frac{2}{3} \gamma \sqrt{\log n} \right) \Delta^R \leq n \exp \left( -\frac{4}{3} \gamma \sqrt{\log n} \right), \tag{3.10}
\]

using \( \Delta^R = \exp[O((\log n)^{1/2 - \delta} \log \log n)] = \exp[o(\sqrt{\log n})] \).

Finally, to complement the above upper bound on \( \mathbb{P}(u \in \xi) \), recall from (3.4) that \( W_T(\xi_{\ell_0}) > \log m - \frac{1}{3} \gamma \sqrt{\log n} \) with probability at least \( 1 - \varepsilon \), from which it follows that \( \mathbb{P}(\xi_{\ell_0} \in S''_{\ell_0}) \geq 1 - 2\varepsilon \), where \( S''_{\ell_0} \subset S'_{\ell_0} \) is the subset of \( S'_{\ell_0} \) from above in which every \( u \in S''_{\ell_0} \) satisfies

\[
\mathbb{P}(\xi_{\ell_0} = u) \leq \frac{1}{m} \exp \left( \frac{1}{4} \gamma \sqrt{\log n} \right) = \frac{1}{n} \exp \left( \frac{\gamma}{3} \sqrt{\log n} \right). \tag{3.11}
\]

### 3.3. Coupling the GW-tree to the random graph.

We now explore the random graph \( G \) and SRW started from a fixed vertex \( v_1 \) in tandem as follows. Upon visiting a vertex for the first time, we expose its neighbors in \( G \) and then proceed to a uniformly chosen one. Our claim is that if \( \tau \) is the first time that the loop-erased trace of this SRW reaches length \( \ell_1 \), then with probability \( 1 - \varepsilon \) we can couple \( (X_1, \ldots, X_\tau) \) to the SRW on the corresponding random GW-tree with the aforementioned truncated exploration process. Indeed, the coupling can only fail in one of the following circumstances:

(a) **Hitting a cycle:** A set \( S \) of size \( s \) vertices in the boundary of the exploration process of \( G \) has at most \( \Delta s \) half-edges that are to be matched in the next round.

A non-tree-edge at a given vertex \( v \in S \) is generated by either

1. matching a pair of its half-edges together — which has probability \( O(\Delta^2/n) \) as long as the number of remaining unmatched half-edges has order \( n \), or
2. pairing one of them to another unmatched half-edge of \( S \) (the more common scenario inducing cycles) which has probability \( O(s\Delta^2/n) \).
By (3.10) we know that \( s \leq n \exp\left(-\frac{1}{4}\gamma \sqrt{\log n}\right) = o(n/\Delta) \) and so indeed the number of unmatched half-edges that remain at time \( \ell_0 \) is of order \( n \). Hence, the probability of encountering such a vertex along some \( t \approx \log n \) trials of exposing the neighborhood of a newly visited vertex has order at most

\[
\frac{st \Delta^2}{n} \lesssim \exp\left(-\frac{1}{4}\gamma \sqrt{\log n} + o\left(\sqrt{\log n}\right)\right) = o(1).
\]

(b) **Hitting a truncation:** By definition, a truncated vertex \( v \) corresponds to a distance-\( R \) ancestor \( u \) for which the event \( B_{\text{tr}}(u) \) holds, and by Lemma 3.1 the probability that the LERW visits such a vertex \( u \) by time \( \ell_0 \) is at most \( \varepsilon \). In the event that the LERW does not visit such a vertex, the SRW must escape from the ray \( \xi \) to distance at least \( R \) in order to hit a truncated vertex, a scenario which has probability at most \( \exp\left(-cR\right) \) thanks to Lemma 2.2; thus, the overall probability of the SRW hitting a truncation by some time \( t \approx \log n \) is at most

\[
t e^{-cR} \lesssim e^{-cR} \log n = o(1)
\]

provided the constant \( c_0 \) in the definition of \( R \) is chosen large enough.

(c) **Mismatched vertex degrees:** While the tree uses a fixed offspring distribution \( Z \), the exploration process in \( G \) dynamically changes as the half-edges are sampled without replacement. Letting \( \tilde{p}_j \) denote the fraction of vertices with \( j \) unmatched half-edges at some point in time, a given half-edge gets matched to a vertex of degree \( j \) with probability \( j \tilde{p}_j / \sum_l l \tilde{p}_l \). Denoting the last variable by \( Z \), and noting that \( \sum_l l \tilde{p}_l \sim \sum_l l p_l \) thanks to (3.10) and the maximum degree bound \( \Delta \), the probability of drawing a different degree than in \( Z \) is given by

\[
\|\mathbb{P}(Z \in \cdot) - \mathbb{P}(Z \in \cdot)\|_{TV} \sim \frac{1}{2} \sum_j \frac{j|p_j - \tilde{p}_j|}{\sum_l l \tilde{p}_l} \lesssim \frac{\Delta}{n} \sum_j |p_j - \tilde{p}_j| n
\]

\[
\lesssim \frac{\Delta}{n} \sum_{i \leq \ell_1} |S_i| \leq \exp\left(-(\frac{1}{4} - o(1))\gamma \sqrt{\log n}\right),
\]

with the last inequality again thanks to (3.10).

### 3.4. Lower bound on mixing in Theorem 2

Set

\[
t^- = \frac{\ell_0}{\nu} = \nu^{-1}\left(\ell_1 - \frac{1}{4d}\gamma \sqrt{\log n}\right),
\]

and observe that (3.6) implies that \( \text{dist}(\rho, X_{t^-}) \leq \ell_1 - (8d)^{-1}\gamma \sqrt{\log n} \) with probability at least \( 1 - \varepsilon \). By Lemma 2.2, applied in a union bound over \( t \leq t^- = O(\log n) \), w.h.p.

\[
\max\{\text{dist}(X_t, \xi) : t \leq t^-\} \leq R \tag{3.12}
\]

for some \( C > 0 \) (depending only on the offspring distribution \( Z \)). It then follows that upon a successful coupling, \( X_{t^-} \) is within distance \( R = O(\log \log n) \) from the set of vertices corresponding to \( \bigcup_{i \leq \ell_1} S_i \), the vertices exposed in \( \ell_1 \) rounds of our truncated exploration process of the GW-tree. Thus, by (3.10) (that holds with probability at least \( 1 - \varepsilon \)), the distribution of \( X_{t^-} \) given a successful coupling has a mass of \( 1 - 2\varepsilon \) on at most \( \Delta R \sum_{i \leq \ell_1} |S_i| = o(n) \) vertices, yielding the required lower bound.
3.5. Upper bound on mixing in Theorem 2.

3.5.1. The case of minimum degree 3 (Theorem 2, Part (i)). Set

\[ t^+ = \frac{\ell_0 + \ell_1}{2\nu} = \nu^{-1} \left( \ell_1 - \frac{1}{16d} \gamma \sqrt{\log n} \right) \]

and note that it follows from (3.6) that with probability at least 1 − ε,

\[ \text{dist}(\rho, X_{t^+}) \geq \ell_0 + \frac{1}{16d} \gamma \sqrt{\log n}, \]

\[ \max_{t \leq t^+} \text{dist}(\rho, X_t) \leq \ell_1 - \frac{1}{16d} \gamma \sqrt{\log n}. \]

On this event, we deduce from the right inequality that a successful coupling of the SRW on G vs. the GW-tree will confine the walk to our tree (and the probability of failing to couple the two processes is at most ε). By the lower bound on dist(\rho, X_{t^+}), w.h.p. the \( \ell_0 \)-prefix of the loop-erased trace of \( \{X_t : t \leq t^+\} \) coincides with the \( \ell_0 \)-prefix of \( \xi \) (or else there would be some point \( t \leq t^+ \) at which dist(\( X_t, \xi \)) \( \gtrsim \sqrt{\log n} \), which is precluded exactly as argued above (3.12)). In particular, for any \( v \) in our GW-tree with some ancestor \( u \in S_{t_0} \) we have \( \mathbb{P}(X_{t^+} = v) \leq \mathbb{P}(\xi_{t_0} = u) \), which is at most \( \frac{1}{n} \exp(\frac{4}{3} \gamma \sqrt{\log n}) \) by (3.11) (that holds with probability at least 1 − 2ε as argued there).

Let \( F_{t^+} \) be the \( \sigma \)-algebra formed by exposing \( \{X_t : t \leq t^+\} \) with its neighborhood in \( G \) as well as the coupled GW-tree to depth max_{\( t \leq t^+ \)} dist(\( X_t, \rho \)) (as defined above), followed by revealing all other edges in \( G \). Let \( S \) be the set of vertices given by

\[ S = \left\{ v : P^{t^+}(v_1, v) \leq \frac{1}{n} \exp\left(\frac{4}{3} \gamma \sqrt{\log n}\right) \right\}; \]

further define \( B \) to be the \( F_{t^+} \)-measurable event consisting of either a failed coupling of \( G \) and the GW-tree, or having dist(\( X_{t^+}, \rho \)) \( \notin (\ell_0, \ell_1) \), or having \( X_{t^+} \notin S \), so that

\[ \mathbb{P}(B) \leq 5\varepsilon, \quad \max_{v \in V} \mathbb{P}(X_{t^+} = v | B^c) \leq \frac{1}{n} \exp\left(\frac{4}{3} \gamma \sqrt{\log n}\right). \]

Henceforth we condition on \( B^c \), and with the fact \( \pi(v) \geq 1/(\Delta n) \) in mind, we infer that

\[ \|\mathbb{P}(X_{t^+} \in \cdot | B^c)/\pi - 1\|_{L^2(\pi)} \leq \exp\left(\frac{4}{3} + o(1)\right) \gamma \sqrt{\log n}. \]

In the setting where \( p_1 = p_2 = 0 \) we can now conclude the proof, using the well-known fact that \( G \) is then w.h.p. an expander and by Cheeger’s inequality the spectral gap of the SRW on \( G \), denoted \( \text{gap} \), is uniformly bounded away from 0 (see [1] on expanders and, e.g., [8, Lemma 3.5] (treating the kernel of the giant component) for the standard argument showing expansion for a graph with said degree sequence). Indeed, if we take

\[ s = c \text{gap}^{-1} \left( \gamma \sqrt{\log n} + \log(1/\varepsilon) \right) \]

for a large enough absolute constant \( c > 0 \), then we will have

\[ \|\mathbb{P}(X_{t^+} \in \cdot | B^c) - \pi\|_{TV} \leq \frac{1}{2} \|\mathbb{P}(X_{t^+} \in \cdot | B^c)/\pi - 1\|_{L^2(\pi)} \leq \varepsilon, \]

as desired.
3.5.2. Allowing degrees 1 and 2 (Theorem 2, Part (ii)). The general case requires a final ingredient, as \( G \) is no longer an expander due to long paths comprised of degree-2 vertices, as well as “heavy” hanging trees. Instead of \( G \), consider the graph \( G' \) on \( n' \leq n \) vertices where every path of degree-2 vertices whose length is larger than \( R = c_0 \log \log n \) is contracted into a single edge, and similarly, every tree whose volume is at least \( R \) is replaced by a single vertex.

Let \( U \) denote the total number of vertices in these contracted structures, and let \( B_{U,t} = \cup_{i \leq t} \{ X_i \in U \} \) denote the event that SRW on \( G \) visits one of these vertices. The fact that \( p_2 \) is uniformly bounded away from 1 implies that \( \mathbb{P}(u \in U) < (\log n)^{-5} \) for any fixed \( u \) provided we choose the constant \( c_0 \) to be sufficiently large (for the long paths this entails \( \mathbb{P}(Y > R) \) where \( Y \sim \text{Geom}(p_2) \), whereas for the hanging trees this corresponds to tail estimates for the size of subcritical GW-trees (see [12])). Hence, for \( t = O(\log n) \) and any fixed initial vertex \( v \), we have \( \mathbb{P}_v(B_{U,t}) \lesssim (\log n)^{-4} \); similarly, if \( V \) is picked according to \( \pi \) then \( \mathbb{P}(\text{deg}(V) = k) \propto kp_k \), so \( \mathbb{E}\text{deg}(V) = \mathbb{E}Z + 1 \) and the set \( A = \{ v : \text{deg}(v) \geq \log^2 n \} \) has \( \pi(A) = O(\log^{-2} n) \) by our assumption \( \mathbb{E}Z \). Hence,

\[
\mathbb{P}_\pi(B_{U,t}) = \sum_v \pi(v) \mathbb{P}_v(B_{U,t}) \leq \pi(A) + \frac{\log^2 n}{n} \sum_{v \notin A} \mathbb{P}_v(B_{U,t}) \lesssim (\log n)^{-2},
\]

and clearly on the event \( B_{U,t}^c \) we can couple the walks on \( G \) and \( G' \). The proof is concluded by noting that the 3-core of \( G' \) is an expander w.h.p., and the effect of expanding some of its edges into paths of length at most \( R \), or hanging trees of volume at most \( R \) on some of its vertices, can only decrease its Cheeger constant to \( c/R \) for some fixed \( c > 0 \). Hence, by Cheeger’s inequality, the spectral gap of \( G' \) satisfies \( \text{gap} > c/R^2 \) for some fixed \( c > 0 \) (depending only on the degrees of \( G' \)), and thereafter taking \( s \) as in (3.13) (noting that \( s \propto \sqrt{\log n(\log \log n)^2} \) now) completes the proof. \( \square \)

The proof will follow from a modification of the argument used for a random graph on a prescribed degree sequence, with the help of the following structure theorem for \( C_1 \).

Theorem 3.2 ([10, Theorem 1]). Let \( C_1 \) be the largest component of \( G(n,p) \) for \( p = \lambda/n \) where \( \lambda > 1 \) is fixed. Let \( \mu < 1 \) be the conjugate of \( \lambda \), i.e., \( \mu e^{-\mu} = \lambda e^{-\lambda} \). Then \( C_1 \) is contiguous to the following model \( C_1^* \):

1. [Kernel] Let \( \Lambda \sim \mathcal{N}(\lambda - \mu, 1/n) \), take \( D_u \sim \text{Poisson}(\Lambda) \) for \( u \in [n] \) i.i.d. and condition that \( \sum D_u 1_{D_u \geq 3} \) is even. Let \( N_k = \#\{ u : D_u = k \} \) and \( N = \sum_{k \geq 3} N_k \).
   Draw a uniform multigraph \( K \) on \( N \) vertices with \( N_k \) vertices of degree \( k \) for \( k \geq 3 \).
2. [2-Core] Replace the edges of \( K \) by paths of i.i.d. \( \text{Geom}(1 - \mu) \) lengths.
3. [Giant] Attach an independent \( \text{Poisson}(\mu) \)-Galton-Watson tree to each vertex.

That is, \( \mathbb{P}(C_1 \in A) \to 0 \) implies \( \mathbb{P}(C_1 \in A) \to 0 \) for any set of graphs \( A \).

The first point is that we develop the neighborhood of a fixed vertex of \( G \) (not of \( C_1 \)) as a GW-tree with offspring distribution \( Z \sim \text{Poisson}(\lambda) \) for \( \lambda > 1 \) fixed. Should its component turn out to be subcritical, we abort the analysis (with nothing to prove).

The second point is that, when analyzing the probability of failing to couple the SRW on \( G \) vs. the GW-tree, mismatched degrees are no longer related to sampling.
with/without replacement from a degree distribution, but rather to the total-variation distance between Poisson(λ) and Bin(n′, λ/n) where n′ is the number of remaining (unvisited) vertices. By (3.10) we know that $n' \geq n (1 - \exp(-\frac{a}{4} \sqrt{\log n}))$, and therefore, by the Stein-Chen method (see, e.g., [17, §6.2]), this total-variation distance is at most $\lambda/n + \|\text{Poisson}(\lambda) - \text{Poisson}(\lambda n'/n)\|_{TV} \leq (\log n)^{-10}$ (with room to spare).

The final and most important point involves the modification of $G$ into $G'$, where the spectral gap is at least of order $\sqrt{\lambda/R}$ the crucial fact that $G'$ has a Cheeger constant of at least $c/R$ follows immediately from that theorem using the expansion properties of the kernel. ■

3.7. The typical distance from the origin. A consequence of our arguments above is the following result on the typical distance of random walk from its origin at $t \asymp \log n$.

**Corollary 3.3.** Consider random walk $(X_t)$ from a fixed vertex $v_1$ in $C_1$ either in the setting of Theorem 1 or of Theorem 2, let $\nu$ denote the speed of random walk on the corresponding Galton-Watson tree, and let $\lambda$ be the mean of its offspring distribution. Then for any fixed $a > 0$, if $t = a \log n$ then w.h.p. $\text{dist}(X_t, v_1) = (1 + o(1)) (\nu a + 1) \log \lambda n$.

Indeed, for $a < \nu^{-1} \log \lambda n$ this follows from coupling the walk to the GW-tree. For $a \geq \nu^{-1} \log \lambda n$, suppose first that $p_1 = p_2 = 0$. As stated in the introduction, it is then the case that the diameter of the graph is $(1 + o(1)) \tilde{D}$ where $\tilde{D} = \log \lambda n$, and a standard concentration estimate shows that once the walk has reached depth $(1 - \varepsilon) \tilde{D}$ in the aforementioned GW-tree then it will never return to level $(1 - 2\varepsilon) \tilde{D}$ within the next (say) $\log^2 n$ steps except with probability $O(n^{-c\varepsilon})$ for some $c > 0$ fixed. Finally, when we allow degrees 1 and 2, the same argument holds as long as the walk avoids the local bottlenecks that differentiate the typical distance in $G$ and its diameter, yet those obstacles are precisely the long paths and large hanging trees treated in §3.5.2, where it was shown that w.h.p. the walk is not incident to any of these by time $t \leq a \log n$.

4. Non-backtracking random walk on random graphs

In this section we consider the NBRW on a random graph $G$ on a degree distribution $(p_i)_{i \geq 0}$ with $p_1 = 0$. Recall that the NBRW is a Markov chain on the set $E$ of directed edges. Its transition probabilities are then simply given as follows: if $e = (u, v)$ and $e' = (v, w)$ are two directed edges such that $w \neq u$, then $P(e, e') = \frac{1}{\delta(u)}$, while the transition probabilities are zero otherwise. Note that our assumption that each vertex has degree at least means that the process is well-defined for all $t$. Then the uniform measure $\pi$ on (directed) edges is invariant. We introduce $P^t(e, e')$ the $t$-step transition probability of NBRW, i.e., $P^t(e, e') = P(X_t = e' | X_0 = e)$ for $e, e' \in E$.

Our assumption on the degree distribution is that

$$\sum_k k(\log k)^2 p_k < \infty. \quad (4.1)$$
Let \( \lambda = \sum_k k p_k < \infty \) (finite under (4.1)) be the mean degree distribution, and let

\[
q_{k-1} = \frac{k p_k}{\lambda}
\]

be the shifted size-biased distribution. Recall that the dimension of harmonic measure for NBRW in the GW-tree with offspring distribution \( q_k \) is

\[
d = \sum_k q_k \log k,
\]

which is finite under (4.1). To state our next result, we let

\[
t^{(\pi)}_{\text{mix}}(\epsilon) = \min \{ t : d^{(\pi)}_{\text{tv}}(t) < \epsilon \}
\]

where \( d^{(\pi)}_{\text{tv}}(t) = \frac{1}{2|E|} \sum_{e \in E} \| P^t(e, \cdot) - \pi \|_{\text{tv}} \), and write \( O_\epsilon(\cdot) \) to stress that the implicit constant in the \( O(\cdot) \)-notation depends on \( \epsilon \).

**Theorem 4.1** (typical starting point). Assume (4.1) and that for a fixed \( \delta > 0 \) the maximum degree in \( G \) is \( \Delta \leq \exp[(\log n)^{1/2-\delta}] \). For any fixed \( 0 < \epsilon < 1 \), w.h.p.

\[
t^{(\pi)}_{\text{mix}}(\epsilon) = \frac{1}{d} \log n + O_\epsilon(\sqrt{\log n}).
\]

**Theorem 4.2** (worst starting point). Assume (4.1), \( p_1 = p_2 = 0 \) and for a fixed \( \delta > 0 \) the maximum degree in \( G \) is \( \Delta \leq \exp[(\log n)^{1/2-\delta}] \). For any fixed \( 0 < \epsilon < 1 \), w.h.p.

\[
t_{\text{mix}}(\epsilon) = \frac{1}{d} \log n + O_\epsilon(\sqrt{\log n}).
\]

**Remark 4.3.** The condition \( p_1 = p_2 \) is necessary for having \( t_{\text{mix}} = O(\log n) \) (e.g., due to long paths, as discussed in §1). The requirement \( \Delta \leq \exp[(\log n)^{1/2-\delta}] \) is not optimal and may be relaxed to weaker moment assumptions on \( (p_k) \) (see, e.g., [2]).

**Remark 4.4.** After completing this work we learned that another proof of Theorem 4.2 was obtained independently by Ben-Hamou and Salez [2] with a weaker assumption on \( \Delta \) and an asymptotic estimate of the total-variation distance within the cutoff window.

### 4.1. Mixing from a typical starting point.

#### 4.1.1. Exploration and good edges. We first suppose that \( G \) is obtained from the configuration model, and reveal the (independent) degrees \( d_w \) of each vertex \( w \in [n] \). Denote \( D_w = d_w - 1 \) (the forward degree). From now on we let

\[
t = \frac{1}{d} \left( \log n + \frac{2K}{\sqrt{d}} \sqrt{\log n} \right),
\]

where \( K \) is fixed positive (and will eventually be chosen large enough). We assume without loss of generality that \( t \) is a nonnegative odd integer and call \( R = \lfloor t/2 \rfloor \). We let \( a_K \) be some fixed real number with the property that \( a_K \to 0 \) as \( K \to \infty \).

Let \( u \in V \) and for a given \( u' \in V \) we suppose that the edge \( e_u = (u, u') \) is present in \( E \). We will expose the neighbourhood structure of \( e_u \) using a breadth-first search procedure, but only when these trees are in some sense relevant for the harmonic measure. We explore the neighbourhood structure of \( e_u \) up to radius \( R \), starting with the neighbours of \( u' \), and then their respective neighbours, and so on up until generation \( R \). We only explore the edges going forward and in particular the resulting subgraph
is always a tree, rooted at the edge \( e_u \). For a vertex \( z \) in the tree being explored, write \( w \preceq z \) to mean that \( w \) is on the path connecting \( u \) to \( z \) within that tree. If during this exploration we reach a vertex \( z \) such that

\[
\left| \sum_{w \preceq z} (\log D_w - d) \right| \geq K\sqrt{R}, \tag{4.4}
\]

then we do not explore any further the connections emanating from \( z \). We call \( T_{e_u} \) the resulting subgraph thus explored, which, as explained, is a tree by construction. Say that the NBRW emanating from \( e_u \) is \emph{good} if it does not leave \( T_{e_u} \) before time \( R \); and say that the edge \( e_u \) is \emph{good} if the NBRW emanating from \( e_u \) is good with probability greater or equal to \( 1 - a_K \). We let \( \mathcal{G} \) be the set of good edges.

Given another edge \( e_v = (v', v) \), we explore the neighbourhood \( T_v \) of the \emph{reverse} edge \( e_v^* = (v, v') \) in exactly the same fashion, except that we also refrain from exploring any potential connections to \( T_u \). Again we shall say that NBRW emanating from \( e_v^* \) is good (with respect to \( e_u \)) if it remains on \( T_v \) up to time \( R \), and say that \( e_v \) is good (with respect to \( e_u \)) if NBRW from \( e_v^* \) is good with probability greater than \( 1 - a_K \). We let \( \mathcal{G}_u \) be the set of good edges (with respect to \( e_u \)).

Let \( \mathcal{F}_0 \) be the \( \sigma \)-field generated by the degree of all vertices. Further let \( \mathcal{F}_u \) be \( \sigma \)-field generated by \( \mathcal{F}_0 \), together with \( T_u \). Likewise, \( \mathcal{F}_{u,v} = \sigma(\mathcal{F}_u \vee \mathcal{F}_v) \). Our first task is to show that \( e_u \) is good w.h.p., and that typically \( e_v \) is also good with respect to \( e_u \).

**Lemma 4.5.** There exists a deterministic sequence \( a_K \to 0 \) such that \( \mathbb{P}(e_u \in \mathcal{G}) \geq 1 - a_K \) and moreover, on an event of probability at least \( 1 - a_K \) we have:

\[
\mathbb{P}(e_u \in \mathcal{G} \mid \mathcal{F}_0) \geq 1 - a_K,
\]

\[
\mathbb{P}(e_u \in \mathcal{G}, e_v \in \mathcal{G}_u \mid \mathcal{F}_0) \geq 1 - a_K.
\]

**Proof.** The same reasoning as in Section 3.3 shows that the NBRW \((X_t)_{t \geq 0}\) can be coupled with probability \( 1 - o(1) \) to a NBRW \((\xi_t)_{t \geq 0}\) on a GW-tree \( T \) (only the events \( (a) \) and \( (c) \), respectively \emph{hitting a cycle} and \emph{mismatched vertex degrees}, are needed).

In this case the sequence of forward degrees \( D_i = (D_{\xi_i}, i = 0, 1, \ldots) \) is an independent sequence of random variables and for \( i \geq 1 \) they are also identically distributed, with distribution given by (4.2). Hence

\[
\mathbb{P}\left( \sup_{k \leq R} \left| \sum_{i=1}^k (\log D_i - d) \right| \geq K\sqrt{R} \right) \leq b_K := \frac{C}{K^2}
\]

by Doob’s maximal inequality, where \( C < \infty \) by (4.1). Since the coupling between the non-bactracking random walk \( X \) on \( G \) and \( \xi \) on the tree \( T \) is successful with probability \( 1 - o(1) \), for \( n \) large enough, the same event holds with \( D_i = D_{\xi_i} \) replaced by \( D_i = D_{X_i} \), with probability at least \( 1 - 2b_K \). But on this event, by definition, \( X \) is good; hence, \( \mathbb{P}(X \text{ bad}) \leq 2b_K \). Recall that the event \( e_u \) is good means \( \mathbb{P}(X \text{ bad \mid G}) \leq a_K \).

\[
\mathbb{P}(e_u \text{ bad}) = \mathbb{E}\left[ \mathbf{1}_{\{\mathbb{P}(X \text{ bad \mid G}) \geq a_K}\right] \leq \mathbb{E}\left[ \frac{\mathbb{P}(X \text{ bad \mid G})}{a_K} \right] = \frac{2b_K}{a_K} \to 0
\]

if \( a_K = \sqrt{2b_K} \). This proves the first part of the lemma. It also follows from Markov’s inequality that \( \mathbb{P}(e_u \in \mathcal{G} \mid \mathcal{F}_0) \geq 1 - a_K \) on an event of probability at least \( 1 - a_K \).
The second part of the lemma follows in much the same way, as the NBRW starting from \( e_v^* \) can be coupled w.h.p. to a NBRW on an independent GW-tree \( T' \).

4.1.2. Estimates on the transition probability from a good edge. We wish to bound the quantity \( P^t(e_u, e_v) \) from below. For \( x \in T_u \) let \( e_x \) be the unique edge in \( T_u \) ending at \( x \). Likewise, for \( y \in T_v \) let \( e_y \) be the edge in \( T_v \) ending at \( y \). We note that

\[
P_t(e_u, e_v) \geq \sum_{x \in T_u, y \in T_v} P^R(e_u, e_x) I_{x,y} \frac{1}{D_x} P^R((x, y), e_y),
\]

where \( I_{x,y} \) denotes the indicator that the edge \((x, y)\) is present in the graph, since one possibility for the NBRW to go from \( u \) to \( v \) in time \( t \) is to choose a vertex \( x \) in \( T_u \) (at distance \( R \)), walk there in time \( R \), require that the edge \((x, y)\) is open, walk to \( y \), and then head for \( v \) in the remaining time \( R \) (since \( t = 2R + 1 \)). Hence, by time-reversibility,

\[
P_t(e_u, e_v) \geq \sum_{x \in T_u, y \in T_v} P^R(e_u, e_x) I_{x,y} \frac{1}{D_x D_y} P^R(e_v^*, e_y). \tag{4.5}
\]

Finally observe that \( P^R_{T_u}(e_u, e_x) \geq P^R_{T_v}(u, x) \) where \( P_T(\cdot, \cdot) \) means that the walk is restricted (not conditioned) to walk within \( T \). Altogether,

\[
P_t(e_u, e_v) \geq Z_{u,v} := \sum_{x \in T_u, y \in T_v} P^R_{T_u}(e_u, e_x) I_{x,y} \frac{1}{D_x D_y} P^R_{T_v}(e_v^*, e_y). \tag{4.6}
\]

We will therefore estimate \( \mathbb{E}[Z_{u,v}] \) and then its variance.

**Lemma 4.6.** Using the notations from Lemma 4.5, if \( e_u \) is good and \( e_v \) is good with respect to \( u \),

\[
\mathbb{E}[Z_{u,v} | F_{u,v}] \geq \frac{1}{2|E|}(1 - 2a_K).
\]

**Proof of Lemma 4.6.** Note that the quantities \( P^R_{T_u}(u, e_x) \), \( P^R_{T_v}(e_v, e_y) \), \( D_x \) and \( D_y \) are all \( F_{u,v} \)-measurable. It remains to compute \( \mathbb{P}((x, y) \text{ open} | F_{u,v}) \). However, given \( F_{u,v} \), the probability that the edge \((x, y)\) is open is given by

\[
\mathbb{P}((x, y) \text{ open} | F_{u,v}) = \frac{D_x D_y}{2(|E| - E(T_u \cup T_v))} \geq \frac{D_x D_y}{2|E|}.
\]

We deduce that

\[
\mathbb{E}[Z_{u,v} | F_{u,v}] \geq \frac{1}{2|E|} \left( \sum_{x \in T_u} P^R_{T_u}(e_u, e_x) \right) \left( \sum_{y \in T_v} P^R_{T_v}(e_v, e_y) \right),
\]

as desired. Now, by Lemma 4.5, if \( e_u \) is good and \( e_v \) is good with respect to \( e_u \), which has probability at least \( 1 - 2a_K \), we know that

\[
\sum_{x \in T_u} P^R_{T_u}(e_u, e_x) \geq \mathbb{P}(X \text{ good} | F_u) \geq 1 - a_K
\]

and likewise \( \sum_{y \in T_v} P^R_{T_v}(e_v, e_y) \geq 1 - a_K \) on \( \{e_v \in \mathcal{G}_u\} \), so \( \mathbb{E}[Z_{u,v} | F_{u,v}] \geq \frac{1}{2|E|}(1 - a_K)^2 \), and the result follows from the fact that \( (1 - a_K)^2 \geq 1 - 2a_K \).
Remark 4.7. Note that the proof of this lemma does not depend on our choice of \( t \) (or equivalently the number of generations of \( T_u \) and \( T_v \)); it is in the following variance computation that \( t \) must be chosen to be appropriately large.

Lemma 4.8. For some constant \( c \) and all \( n \) sufficiently large,
\[
\text{Var}(Z_{u,v} \mid \mathcal{F}_{u,v}) \leq \frac{1}{cn^2} \exp(-cK\sqrt{\log n}).
\]

Proof. Clearly, we can write
\[
\text{Var}(Z_{u,v} \mid \mathcal{F}_{u,v}) \leq \sum_{x,x' \in T_u} \sum_{y,y' \in T_v} P^R_{T_u}(e_u, e_x) P^R_{T_v}(e_v, e_{x'}) \times \frac{1}{D_x D_x' D_y D_y'} \text{Cov}(I_{x,y}, I_{x',y'} \mid \mathcal{F}_{u,v}) P^R_{T_u}(e_v, e_y) P^R_{T_v}(e_v, e_{y'}).\]

Now, if \( x \neq x' \) or \( y \neq y' \), it is easy to see that
\[
\text{Cov}(I_{x,y}, I_{x',y'} \mid \mathcal{F}_{u,v}) = \mathbb{P}(x \sim y \mid \mathcal{F}_{u,v}) \left[ \mathbb{P}(x' \sim y' \mid x \sim y, \mathcal{F}_{u,v}) - \mathbb{P}(x' \sim y' \mid \mathcal{F}_{u,v}) \right] \leq \frac{D_x D_y}{2(|E| - E(T_u \cup T_v))} \left[ \frac{D_{x'} D_{y'}}{2(|E| - E(T_u \cup T_v) - 1)} - \frac{D_{x'} D_{y'}}{2(|E| - E(T_u \cup T_v))} \right] \leq \frac{D_x D_y D_{x'} D_{y'}}{2|E|^3} (1 + o(1)),
\]
where in the last line, we have used that because of the way \( T_u \) and \( T_v \) were constructed, \(|E(T_u)| = o(n)\). To see this, observe that by definition, at each leaf \( x \) of the tree \( T_{e_u} \),
\[
\mathbb{P}(X \text{ goes through } x) = \prod_{\omega \leq x} (1/D_w) \geq e^{-dR - K\sqrt{R}} \geq n^{-1/2} e^{-O(\sqrt{\log n})}. \tag{4.7}
\]
Since these events are disjoint and the depth of the tree is at most \( R = O(\log n) \), the total number of edges revealed in \( T_u \) is at most \( \Delta R\sqrt{n}e^{O(\sqrt{\log n})} = \sqrt{n}e^{O(\sqrt{\log n})} = o(n) \).

We now consider the diagonal terms where \( x = x' \) and \( y = y' \). In this case
\[
\text{Var}(I_{x,y} \mid \mathcal{F}_{u,v}) \leq \mathbb{E}[I_{x,y} \mid \mathcal{F}_{u,v}] \leq \frac{D_x D_y}{2|E|} (1 - o(1)).
\]
Putting things together and noting the various cancellations, we get, since \(|E| \geq n\) under our assumptions,
\[
\frac{\text{Var}(Z_{u,v} \mid \mathcal{F}_{u,v})}{\deg(v)^2} \leq \frac{1 + o(1)}{2|E|^3} \sum_{(x,y,\neq (x',y')} \sum_{x \in T_u, y \in T_v} P^R_{T_u}(e_u, e_x) P^R_{T_v}(e_v, e_{x'}) P^R_{T_u}(e_u, e_y) P^R_{T_v}(e_v, e_{y'}) \times \frac{1}{D_x D_y} \left[ \max_{x \in T_u} P^R_{T_u}(e_u, e_x) \right] \left( \max_{y \in T_v} P^R_{T_v}(e_v, e_y) \right).
\]
Now, if \( x \in T_u \), reasoning as in (4.7),
\[
P^R_{T_u}(e_u, e_x) \leq \exp\left(-dR_u + K\sqrt{R}\right) \leq n^{-1/2} \exp(-cK\sqrt{\log n}) \tag{4.8}
\]
for $n$ sufficiently large. Consequently,

$$\text{Var} \left( Z_{u,v} \mid F_{u,v} \right) \leq \frac{1 + o(1)}{c n^3} + \frac{1 + o(1)}{c n^2} \exp(-cK\sqrt{\log n}),$$

and the result follows. ■

4.1.3. Upper bound on the mixing time in Theorem 4.1. By Chebyshev’s inequality, we deduce from Lemmas 4.6 and 4.8 that on \( \{ e_u \in G \} \cap \{ e_v \in G_u \} \),

$$\mathbb{P} \left( Z_{u,v} \leq (1 - 4aK) \frac{1}{2|E|} \right) \leq \mathbb{Var} \left( Z_{u,v} \mid F_{u,v} \right) \leq \frac{1}{(2 \deg(v) a_K/2|E|)^2} \leq c^{-1} \exp(-cK\sqrt{\log n}) \to 0. \quad (4.9)$$

We call $H_{u,v}$ the complement of the event in the left hand side, i.e.,

$$H_{u,v} = \left\{ Z_{u,v} \geq (1 - 4aK) \frac{1}{2|E|} \right\}.$$

Then

$$P^t(e_u, e_v) \geq (1 - 4aK) \frac{1}{2|E|} \quad (4.10)$$

if $e_u$ is good, $e_v$ is good with respect to $e_u$, and $H_{uv}$ holds. Therefore,

$$d^{(\pi)}(t) \leq 4aK + \sum_{e_v} \pi(e_v) 1_{\{ e_v \notin \mathcal{G}_u, H_{uv} \}} + \sum_{e_v} \pi(e_v) 1_{\{ e_v \in \mathcal{G}_u \}}.$$

We call $\varepsilon_1$ and $\varepsilon_2$ respectively the last two error terms on the right hand side above. Note that

$$\mathbb{E}[\varepsilon_1 \mid F_0] \leq \sum_{v, v' \in V} \mathbb{P}(e_v = (v, v') \in E \mid F_0) \frac{1}{2|E|} \mathbb{P}(e_v \in \mathcal{G}_u, H_{uv}^c \mid F_0, e_v \in E) \to 0$$

by (4.9) since the right hand side of (4.9) is non random. Likewise

$$\mathbb{E}[\varepsilon_2 \mid F_0] \leq \sum_{e_v} \pi(e_v) \mathbb{P}(e_v \notin \mathcal{G}_u \mid F_0) \leq a_K$$

by Lemma 4.5. Consequently,

$$\mathbb{E} \left[ d^{(\pi)}(t) \mid F_0 \right] \leq 6aK$$

for $n$ sufficiently large. By Markov’s inequality (and as $d^{(\pi)}(t) \leq 1$), $d^{(\pi)}(t) \leq \sqrt{6a_K}$ with probability at least $1 - \sqrt{2a_K}$ for $n$ sufficiently large. Theorem 4.1 follows. ■

4.1.4. Lower bound on the mixing time in Theorem 4.1. Let

$$t^- = \frac{1}{d} \left( \log n - \frac{2K}{\sqrt{d}} \sqrt{\log n} \right),$$

where $K$ is fixed positive (and will eventually be chosen large enough). Again we may assume that $t^-$ is odd and we let $R^- = \lfloor t/2 \rfloor$. 
We consider the analogous trees $T_u^-$ and $T_v^-$ defined by exploring the neighborhoods of $e_u$ and $e_v^*$ resp., but only to distance $R^-$, while the exploration only proceeds if
\[
\left| \sum_{w \leq z} (\log D_w - d) \right| \geq K \sqrt{n} R^-,
\]
otherwise it is truncated. Since $T_u^- \subset T_u$ and $t^- \leq t$ we know from Lemma 4.5 that with probability at least $1 - a_K$, a NBRW $X$ starting from $e_u$ stays on $T_u^-$. Observe that if $x$ is any leaf of $T_u^-$ then by (4.11),
\[
\mathbb{P}(X \text{ goes through } e_x) \geq e^{-dR^- - K \sqrt{n} R^-} \geq \sqrt{n} e^{O(\sqrt{\log n})},
\]
so that there are at most $\sqrt{n} e^{-O(\sqrt{\log n})}$ leaves in $T_u^-$. Since the maximal degree is $\Delta = e^{O(\sqrt{\log n})}$ this means that there are at most $\sqrt{n} e^{-O(\sqrt{\log n})}$ half-edges leaving $T_u$. Likewise there are at most $\sqrt{n} e^{-O(\sqrt{\log n})}$ half-edges leaving $T_v^-$. Therefore the probability that any of these half-edges is paired is at most $\left( \sqrt{n} e^{-O(\sqrt{\log n})} \right)^2 / |E|$. (Here $E$ is the number of edges not revealed in either of these trees and is hence at least $n$.) Thus, this probability is at most $e^{-O(\sqrt{\log n})} \rightarrow 0$. Altogether, for $X_{t^-}$ to be at $e_v$, one of three unlikely events would have to occur: $X$ leaves $T^-_u$ before $R^-$, or $T^-_u$ and $T^-_v$ are connected by an edge, or $X^*$ leaves $T^-_v$ before $R^-$. The sum of these probabilities is at most $2a_K + o(1) \leq 3a_K$. Since $e_v$ was an arbitrary good edge, the result follows.

**Remark 4.9.** Since this lower bound holds for a typical starting edge, a fortiori it holds for the worst edge. Hence this also provides the lower bound required for Theorem 4.2.

4.2. Mixing from a worst starting point with minimum degree 3.

4.2.1. Concentration inequality. Our first task is to get a sharper inequality than (4.9) controlling the fluctuations of the variable $Z_{u,v}$ when $e_u \in \mathcal{G}$ and $e_v \in \mathcal{G}_u$.

**Lemma 4.10.** Suppose $e_u \in \mathcal{G}$ and $e_v \in \mathcal{G}_u$. Then
\[
\mathbb{P}\left( Z_{u,v} \leq (1 - 4a_K) \frac{1}{2|E|} \right| \mathcal{F}_{u,v} ) \leq \gamma_n \quad \text{where} \quad \gamma_n = 2 \exp\left[ -Ca_K^2 e^{K \sqrt{\log n} / \Delta^2} \right].
\]
In particular, $\gamma_n$ decays faster than any polynomial.

**Proof.** As we will see this will be a consequence of martingale estimates due to Freedman [15] which we recall for convenience. Let $(\mathcal{M}_t)_{t \geq 1}$ denote a sequence of martingale differences, with respect to some filtration $(\mathcal{E}_t)$, so that $S_n = \sum_{t \leq n} \mathcal{M}_t$ forms a martingale started at 0. Suppose that the increments $|\mathcal{M}_t|$ are bounded by 1. Let $V_t = \text{Var}(\mathcal{M}_t | \mathcal{E}_{t-1})$ denote the conditional variance, and for some fixed $N \geq 1$, let $V = \sum_{t \leq N} V_t$. Then (a corollary of) Theorem (1.6) in Freedman implies that
\[
\mathbb{P}(S_n \geq a, V_n \leq b) \leq \exp\left[ -\frac{a^2}{2(a + b)} \right].
\]
If instead of being bounded by 1, the increments are bounded by some constant $\gamma$, (i.e., if we assume $|\mathcal{M}_t| \leq \gamma$) then this becomes:
\[
\mathbb{P}(S_n \geq a, V_n \leq b) \leq \exp\left[ -\frac{a^2}{2(a \gamma + b)} \right].
\]

(4.12)
Let $\mathbb{P}^*$ denote the conditional probability $\mathbb{P}^*(\cdot) = \mathbb{P}(\cdot \mid \mathcal{F}_{u,v})$. Suppose we reveal the edges of $T_u$ and $T_v$ at distance $R$ from $e_u$ and $e_v$ respectively in some arbitrary order, and call $x_1, \ldots, x_m$ and $y_1, \ldots, y_k$ their respective endpoints. We first examine $x_1$ and see if it is connected to $y_1, \ldots, y_k$, then move on to $x_2$ and so on until we examine whether $x_m$ is connected to any of $y_1, \ldots, y_k$; thus, there are $mk$ steps altogether. For $1 \leq \ell \leq mk$, we let $\mathcal{E}_\ell$ be the $\sigma$-algebra generated by $\mathcal{F}_{u,v}$ and the connections revealed during the first $\ell$ steps of this procedure, so $(\mathcal{E}_\ell)_{1 \leq \ell \leq mk}$ forms a filtration.

For $1 \leq i \leq m$ and $1 \leq j \leq k$, let $\ell = (i - 1)m + j$ and let

$$Y_{ij} = \mathbb{E}\left(P_{T_u}^R(e_u, e_{x_i})I_{x_i, y_j} \frac{1}{D_{x_i, y_j}} P_{T_v}^R(e_v, e_{y_j}) \mid \mathcal{E}_\ell\right).$$

Note that

$$Y_{ij} \leq \frac{1}{2|E|} P_{T_u}^R(e_u, e_{x_i}) P_{T_v}^R(e_v, e_{y_j}), \quad (4.13)$$

$\mathbb{P}^*$-a.s., and

$$\mathbb{P}^*(\cdot) = \mathbb{P}(\cdot \mid \mathcal{F}_{u,v})$$

\[Y_{ij} \leq \frac{1}{2|E|} P_{T_u}^R(e_u, e_{x_i}) P_{T_v}^R(e_v, e_{y_j}), \quad (4.14)\]

where the $o(1)$-term is nonrandom and uniform over $1 \leq i \leq m, 1 \leq j \leq k$.

Moreover, if we let

$$M_\ell = P_{T_u}^R(e_u, e_{x_i})I_{x_i, y_j} \frac{1}{D_{x_i, y_j}} P_{T_v}^R(e_v, e_{y_j}) - Y_{ij},$$

$$S_\ell = \sum_{\ell' \leq \ell} M_{\ell'},$$

then $(S_\ell)_{1 \leq \ell \leq mk}$ is a $(\mathbb{P}^*, \mathcal{E})$-martingale. Furthermore, its increments are bounded by

$$|M_\ell| \leq C \max_{1 \leq i \leq m, 1 \leq j \leq k} P_{T_u}^R(e_u, e_{x_i}) P_{T_v}^R(e_v, e_{y_j}) \leq \frac{C}{n} e^{-cK\sqrt{\log n}},$$

by (4.8). Let $V_\ell = \text{Var}(M_\ell \mid \mathcal{E}_{\ell-1})$ denote the conditional variance. Observe that

$$V_\ell \leq \left|P_{T_u}^R(e_u, e_{x_i}) P_{T_v}^R(e_v, e_{y_j})\right|^2 \frac{1}{2|E|} (1 + o(1)),$$

so that

$$V := \sum_{1 \leq \ell \leq mk} V_\ell \leq \frac{1}{2|E|} (1 + o(1)) \sum_{i,j} \left|P_{T_u}^R(e_u, e_{x_i}) P_{T_v}^R(e_v, e_{y_j})\right|^2$$

$$\leq \frac{C}{n} \sum_{1 \leq i \leq m, 1 \leq j \leq k} P_{T_u}^R(e_u, e_{x_i}) P_{T_v}^R(e_v, e_{y_j}) \leq \frac{C}{n^2} e^{-cK\sqrt{\log n}},$$

$\mathbb{P}^*$-a.s. We deduce from (4.12) and the fact $n \leq |E| \leq 2\Delta n$ under our assumptions that

$$\mathbb{P}^* \left(|S_{mk}| \geq \frac{2aK}{|E|}\right) = \mathbb{P}^* \left(|S_{mk}| \geq \frac{2aK}{|E|}, V \leq \frac{C}{n^2} e^{-cK\sqrt{\log n}}\right)$$

$$\leq 2 \exp \left[ -\frac{1}{2} \left(\frac{aK}{n} (C/n) e^{-cK\sqrt{\log n}} + 2(C/n^2) e^{-cK\sqrt{\log n}}\right)^2 \right]$$

$$\leq 2 \exp \left[ -CaK^2 e^{cK\sqrt{\log n}/\Delta^2} \right]. \quad (4.15)$$
Now, from (4.13),
\[ \sum_{i,j} Y_{ij} \geq \frac{1}{2|E|} \mathbb{P}_u(\tau_u > R_u) \mathbb{P}_v(\tau_v > R_v) \geq \frac{1}{2|E|} (1 - 2a_K); \]
hence,
\[ \mathbb{P}^*(Z_{u,v} \leq (1 - 4a_K) \frac{d_v}{2|E|}) \leq \mathbb{P}^*\left(|S_{mk}| \geq \frac{2a_K}{2|E|}\right), \]
and the desired result thus follows from (4.15). \[\blacksquare\]

4.2.2. Reaching good edges after a short delay. We now show that after a small amount of time (namely, \(r = c_1 \log \log n\) for some sufficiently large constant \(c_1 > 0\)) the walk will arrive at a good edge with probability at least \(1 - 2a_K\) over walks, uniformly over all starting points (with high probability over graphs as \(n \to \infty\)).

**Lemma 4.11.** There exists a constant \(c_1 > 0\) such that if we set \(r = c_1 \log \log n\), then the following holds w.h.p.: for any edge \(e_u \in E\), \(P^r(e_u, S) \geq 1 - 2a_K\).

**Proof.** The proof follows the idea of Lemma 4.5 but with a few modifications. First, we explore the entire neighbourhood \(S_r\) of \(e_u\) up to distance \(r\). Doing so we reveal at most \(|S_r| \leq \Delta^r = \exp[o(\sqrt{\log n})]\) edges. Let \(e_1, \ldots, e_s\) denote the edges at distance \(r\) in \(S_r\) from \(e_u\). We shall see that each of the edges \(e_i\) are good with probability at least \(1 - a_K\), independently of each other. To check this, suppose we fix \(1 \leq i \leq s\) and reveal the trees \(T_{e_1}, \ldots, T_{e_{i-1}}\) arising in the definition of the edge \(e_i\) being good (see Section 4.1.1). We now consider a NBRW \(X^i\) started from \(e_i\), run for a duration \(R\). We claim that, conditionally on \(H_{i-1}\) which contains \(S_r\) and the trees \(T_1, \ldots, T_{i-1}\), \(X^i\) can be coupled to an independent non-backtracking random walk \(\xi^i\) on an independent GW-tree \(T^i\) with offspring distribution given by (4.2). The key for doing this is to only reveal the graph adjacent to the path of each of the \(X^i\).

Note that \(S_r\) itself contains at most \(\Delta^r = \exp[o(\sqrt{\log n})]\) edges. Also, each of the tree \(T_{e_i}\) contains relatively few edges. This is because by definition, at each leaf \(v\) of the tree \(T_{e_i}\), \(P(X^i \text{ goes through } v) \geq e^{-dR - K\sqrt{R}} \geq n^{-1/2}e^{-O(\sqrt{\log n})}\). Since these events are disjoint and the depth of the tree is at most \(R = O(\log n)\), the total number of edges revealed in \(T_{e_i}\) is at most \(\Delta R \sqrt{n} e^{O(\sqrt{\log n})} = \sqrt{n} e^{O(\sqrt{\log n})}\). When we sum over all trees \(T_{e_1}, \ldots, T_{e_s}\) we get at most \(\sqrt{n} e^{O(\sqrt{\log n})}\) again since \(s \leq |S_r| \leq e^{O(\sqrt{\log n})}\).

Hence at each stage in the exploration of the part of the tree \(T^i\) which is adjacent to \(X^i\) we have a total of \(|E| \geq n\) half-edges which have not been matched at any point in the exploration so far (as the degree of each vertex is deterministically at least 3). Since the degree of each vertex is at most \(\Delta\) and there are exactly \(R\) steps in the exploration of that part of the tree, the probability of hitting a cycle is therefore at most \(\Delta R \times \sqrt{n} e^{O(\sqrt{\log n})} / |E| \leq n^{-1/2}e^{O(\sqrt{\log n})}\).

Likewise, when accounting for the mismatch in the degree distribution, if \(\tilde{V}_j\) is the number of vertices with \(j\) available (not currently matched) half-edges attached to it, then a given half-edge gets matched to a vertex of degree \(j\) with probability \(j \tilde{V}_j / \sum \ell \tilde{V}_\ell\).
So, the probability of drawing a different degree than the offspring $Z$ in (4.2) is at most

$$\|\mathbb{P}(\tilde{Z} \in \cdot) - \mathbb{P}(Z \in \cdot)\|_{tv} = \frac{1}{n} \sum_{j} \frac{|V_j - \tilde{V}_j|}{\sum_{e} \ell_{V_e}} \leq \frac{\Delta}{n} \sum_{j} |V_j - \tilde{V}_j| \leq \frac{\Delta}{n} \sqrt{n} e^{O(\sqrt{\log n})}.$$ 

Even when summed over $R$ steps of the exploration procedure this still tends to 0.

Now observe that by Lemma 4.5, $\xi^i$ is good with probability at least $1 - b_K$ where $b_K \to 0$ (when averaged over both the walk and the tree $T^i$; this was a straightforward consequence of Doob’s maximal inequality). Moreover, given $S_r$ and $\mathcal{H}_{i-1}$, the above argument shows that $X^i$ will be good with probability at least $1 - b_K - o(1) \geq 1 - 2b_K$, where the $o(1)$ accounts for the probability that the coupling with an independent walk on an independent tree fails. We deduce that

$$\mathbb{P}(e_i \text{ bad } \mid \mathcal{H}_{i-1}) = \mathbb{E} \left[ 1_{\{\mathbb{P}(X^i \text{ bad } \mid G) \geq a_K \}} \mid \mathcal{H}_{i-1} \right] \leq \mathbb{E} \left[ \frac{\mathbb{P}(X^i \text{ bad } \mid G)}{a_K} \mid \mathcal{H}_{i-1} \right] \leq a_K^{-1} \mathbb{P}(X^i \text{ bad } \mid \mathcal{H}_{i-1}) \leq 2b_K/a_K = a_K$$

if $a_K = \sqrt{2b_K} \to 0$.

Let $\mu_i = P^r(e_u, e_i)$ so $\sum_{i=1}^{s} \mu_i \leq 1$ (where the $\leq$ accounts for the fact that in the unlikely event there is a cycle, the NBRW might be at distance less than $r$ at time $r$). Let $X = \sum_{i=1}^{s} \mu_i 1_{\{e_i \text{ bad} \}}$, so that $X$ is the fraction of bad edges (averaged with respect to harmonic measure at distance $r$). Then by Markov’s inequality, for all $\lambda > 0$,

$$\mathbb{P}(X \geq 2a_K \mid S_r) \leq e^{-2a_K \lambda} \mathbb{E}(e^{\lambda X} \mid S_r) \leq e^{-2\lambda a_K} \prod_{i=1}^{s} (1 + a_K (e^{\lambda \mu_i} - 1)) \leq e^{-2\lambda a_K} \exp \left( a_K \sum_{i=1}^{s} (e^{\lambda \mu_i} - 1) \right).$$

Take $\lambda = 1/ \max_i \mu(e_i)$ and note that for $0 \leq x \leq 1$, we have $0 \leq e^x - 1 \leq (e-1)x$ by convexity of the exponential function, hence

$$\mathbb{P}(X \geq 2a_K \mid S_r) \leq \exp \left( -2a_K \lambda + a_K (e - 1) \right) \leq \exp(a_K (e - 3)).$$

Now, since $p_1 = p_2 = 0$, $\max_i \mu(e_i) \leq 2^{-r}$, and hence $\lambda \geq 2^r$. Therefore

$$\mathbb{P}(X \geq 2a_K \mid S_r) \leq \exp(-e^{2^r}).$$

Since $r = c_1 \log \log n$ and $c_1$ may be chosen large enough that $2^r \geq (\log n)^2$, it follows that $\mathbb{P}(X \geq 2a_K \mid S_r) = o(n^{-2})$. Say that $e_u$ is very bad if $\mathbb{P}(X \geq 2a_K \mid G) \geq n^{-2}$ where $X = X(e_u)$ is the random variable above. Then

$$\mathbb{P}(e_u \text{ very bad}) = \mathbb{E} \left[ 1_{\{\mathbb{P}(X \geq 2a_K \mid G) \geq n^{-2} \}} \right] \leq \mathbb{E} \left[ \frac{\mathbb{P}(X \geq 2a_K \mid G)}{n^{-2}} \right] = o(n^{-2})$$

by the above. Hence w.h.p. there exists no very bad edge in $G$, and consequently, w.h.p. every edge $e_u$ has $P^r(e_u, \emptyset) \geq 1 - 2a_K$. The result follows. ■
4.2.3. Proof of Theorem 4.2. As before, define
\[ H_{u,v} = \left\{ Z_{u,v} \geq (1 - 4aK) \frac{1}{2|E|} \right\}. \]

Then we know from Lemma 4.10 that \( \mathbb{P}(H_{u,v}^c | F_{u,v}) \leq \gamma_n \) and hence taking expectations and using a union bound,
\[ \mathbb{P} \left( \bigcup_{u,v} \{e_u \in \mathfrak{G} \} \cap \{e_v \in \mathfrak{G}_u \} \cap H_{u,v}^c \right) \leq n^2 \gamma_n \rightarrow 0, \]
from which it follows that the event \( \Gamma := \bigcap \{H_{u,v} : e_u \in \mathfrak{G}, e_v \in \mathfrak{G}_u\} \) occurs w.h.p. Suppose that indeed \( \Gamma \) holds and let \( e_u \in \mathfrak{G}, e_v \in \mathfrak{G}_u \). Then
\[ P^t(e_u, e_v) \geq Z_{u,v} \geq (1 - 4aK) \pi(e_v) \]
by definition of \( Z_{u,v} \) and \( H_{u,v} \). Hence, if \( \Gamma \) holds, for \( e_u \in \mathfrak{G} \),
\[ \|P^t(e_u, \cdot) - \pi\| = \sum_{e_v} (\pi(e_v) - P^t(e_u, e_v))_+ \leq \pi(\mathfrak{G}_u^c) + \sum_{e_u \in \mathfrak{G}_u} 4aK \pi(e_v) \leq 5aK \quad (4.16) \]
by Lemma 4.5. Hence applying the Markov property of NBRW, if \( e_u \) is arbitrary (and we still assume \( \Gamma \) holds),
\[ \|P^{t+r}(e_u, \cdot) - \pi\| \leq (1 - P^r(e_u, \mathfrak{G})) + \sup_{e_w \in \mathfrak{G}} \|P^t(e_w, \cdot) - \pi\| \leq 7aK \]
by Lemma 4.11 and (4.16). Since \( u \) is arbitrary, we deduce that on the event \( \Gamma \) (which occurs w.h.p. as argued above) we have \( d_{tv}(t + r) \leq 7aK \), and the result follows. ■

References


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