Quantum Rate Distortion Coding with Auxiliary Resources

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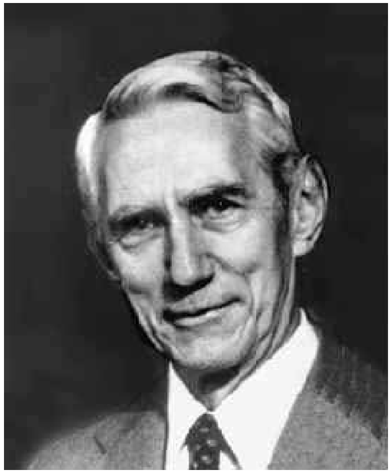
arXiv:1108.4940, **arXiv:1210.6962**, **arXiv:1212.5316**

Beyond IID in Information Theory,
University of Cambridge, Cambridge, UK,
January 11, 2013

Classical Rate Distortion Coding

Classical rate distortion coding is
the theory of **lossy data compression**



Recall Shannon's **noiseless coding theorem**:

The entropy *H*(*X*) of an IID information source
is the fundamental limit to data compression

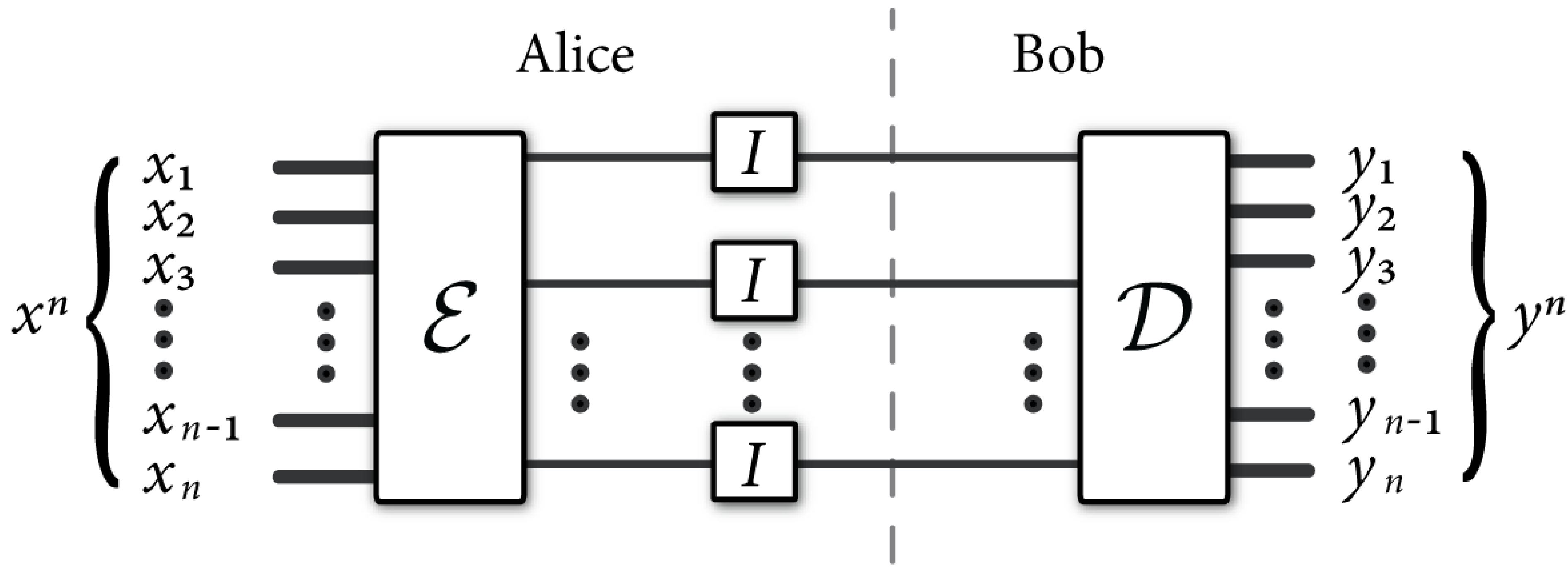
Is there a meaningful way for us
to compress data *below* the Shannon limit?

There can't be an IID rate-error trade-off (*strong converse to Shannon's theorem*),

But well-known that there can be a **rate vs. distortion trade-off**

Claude E. Shannon. Coding theorems for a discrete source with a fidelity criterion.
IRE International Convention Records, 7:142–163, 1959.

Classical Rate Distortion Coding



Distortion between input x^n and output y^n is measured on average:

$$\overline{d}(x^n, y^n) \equiv \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i)$$

Single symbol distortion measure *d* could be Hamming distance, squared error, etc.

Claude E. Shannon. Coding theorems for a discrete source with a fidelity criterion. *IRE International Convention Records*, 7:142–163, 1959.

Shannon's Rate Distortion Theorem

Achievability: For any R, $D \geq 0$, the pair (R,D) is an **achievable rate-distortion pair** if there exists a sequence of (n,R) rate-distortion codes such that

$$\lim_{n\to\infty} \mathbb{E}_{X^n}\{\bar{d}(X^n, (\mathcal{D}^n \circ \mathcal{E}^n)(X^n))\} \leq D$$

Rate Distortion Function:

$$R(D) \equiv \inf\{R : (R, D) \text{ is achievable}\}$$

Convexity in *D*: For linear distortion measure, $R(D)$ is convex in D.
(*seen operationally by a simple time-sharing argument*)

Shannon's Rate Distortion Theorem:

$$R(D) = \min_{p_{Y|X}(y|x)\,:\,\mathbb{E}_{X,Y}\{d(X,Y)\}\leq D} I(X;Y)$$

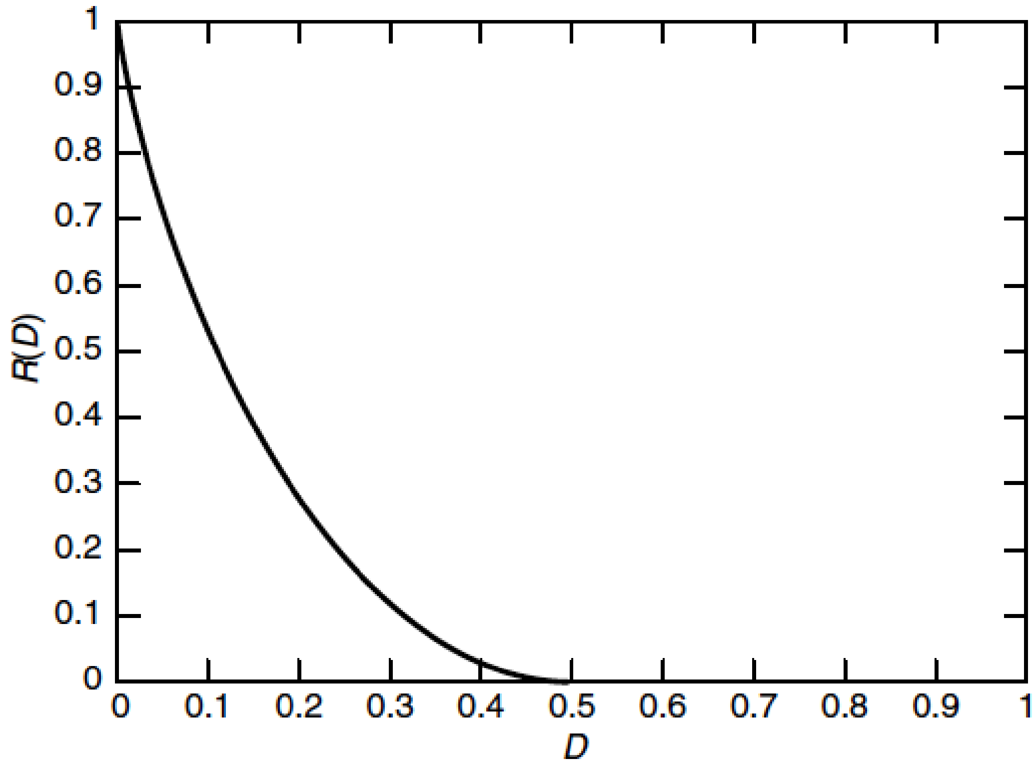
Claude E. Shannon. Coding theorems for a discrete source with a fidelity criterion. *IRE International Convention Records*, 7:142–163, 1959.

Simple Example: Random Bit Source

Suppose that the source is a random bit: Pr{X = 0} = Pr{X = 1} = 1/2

Then the rate-distortion function (for Hamming distortion) is given by

$$R(D) = \begin{cases} 1 - h_2(D) & \text{if } 0 \le D \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < D \le 1. \end{cases}$$



Thomas Cover and Joy Thomas. *Elements of Information Theory*

Why develop Quantum Rate Distortion?

Schumacher compression gives that the von Neumann entropy is the ultimate limit of compressibility for an IID quantum info. source

We would like to have a meaningful way to compress quantum data at a rate lower than the Schumacher limit

(to develop a quantum analog of the rate vs. distortion trade-off. Also, the strong converse for Schumacher compression implies that there is no IID rate-error trade-off)

A theory of lossy quantum data compression should provide an answer to this fundamental question

Could be helpful in quantizing CV quantum systems to qubits

"*One reason for studying it is that rate distortion theory is one of the basic fundamental results for classical Shannon theory, and if we do not study the quantum version, this leaves a noticeable hole in quantum information theory.*" — QIP 2012 Program Committee

Timeline of Quantum Rate Distortion

1998 — Barnum introduces quantum rate distortion:
1) develops definitions and task to be accomplished.
2) conjectures incorrectly that coherent information should be relevant for QRD

2000 — Devetak and Berger study classically-assisted QRD and give a single-letter expression for CA QRD function for isotropic qubit source and entanglement fidelity distortion.

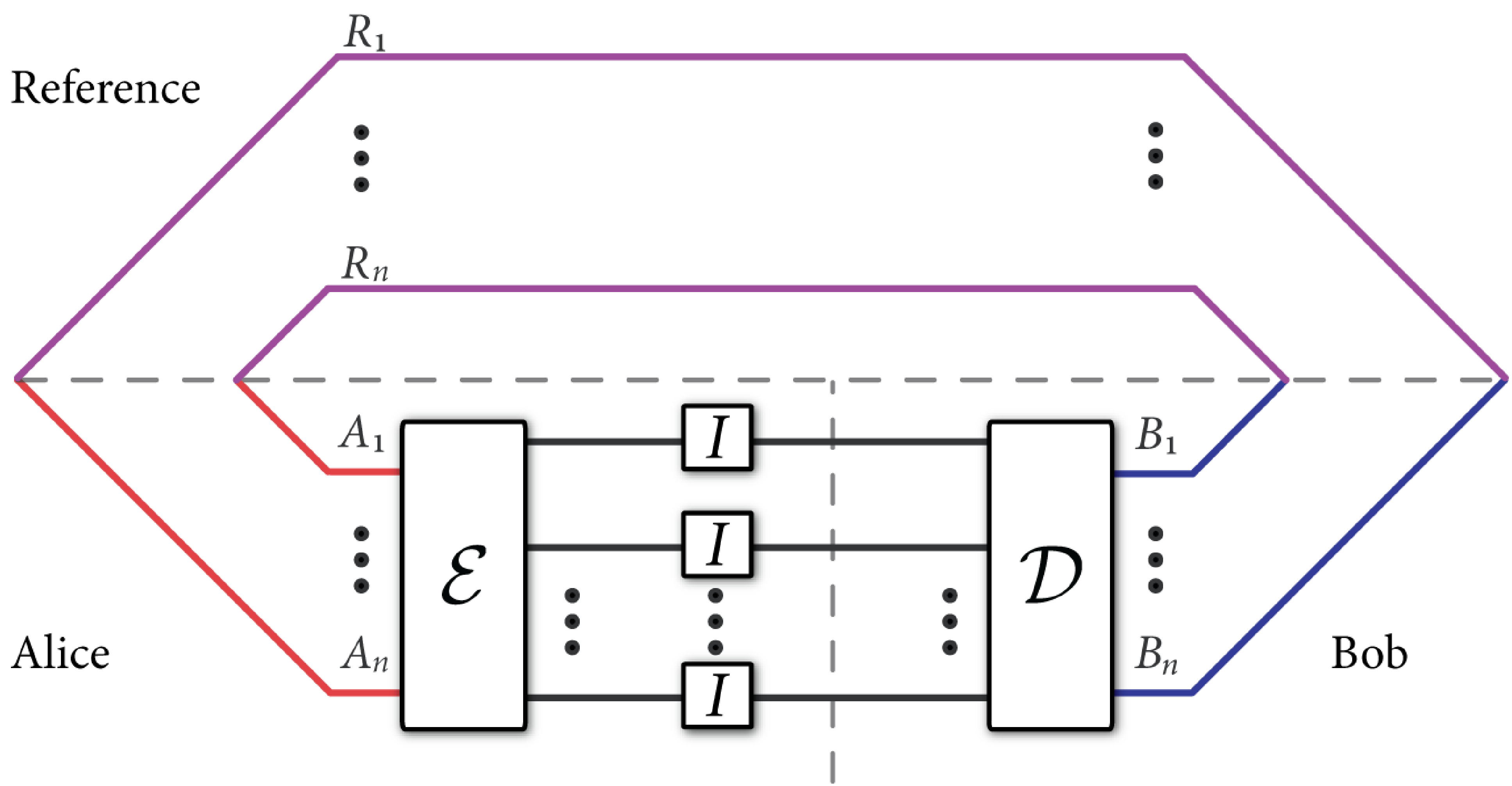
2001 — Winter and Ahlswede introduce the idea of a distortion observable as a framework for generalizing classical rate distortion theory (unpublished manuscript)

2002 — Winter makes the observation that a reverse Shannon theorem implies a rate distortion theorem (though see Verdu Steinberg 1996)

2006 — Luo and Devetak study simulation of a classical channel with quantum side information (QSI) and generalize the Wyner-Ziv result to the case of QSI using Winter's observation

2008 — Thrown off by the Barnum conjecture, Chen and Wang study the quantization of a quantum Gaussian source and evaluate the coherent information lower bound

Quantum Rate Distortion Coding Task



Initial state is $\left(|\psi^\rho\rangle_{RA}\right)^{\otimes n}$, a state which purifies $\left(\rho_A\right)^{\otimes n}$

Howard Barnum. Quantum Rate-Distortion Coding. Phys. Rev. A 62, 042309 (2000)

Distortion Measure

Average distortion between input state and output state is

$$\overline{d}(\rho^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n) \equiv \frac{1}{n}\sum_{i=1}^{n} d(\rho, \mathcal{F}_n^{(i)}),$$

where the *i*th marginal operation is defined as

$$\mathcal{F}_n^{(i)}(\xi) \equiv \mathrm{Tr}_{A_1 A_2 \cdots A_{i-1} A_{i+1} \cdots A_n}[\mathcal{F}_n(\rho^{\otimes(i-1)} \otimes \xi \otimes \rho^{\otimes(n-i)})]$$

and a natural choice for the single-symbol distortion *d* arises from the **entanglement fidelity**:

$$d(\rho, \mathcal{N}) = 1 - \langle \psi^{\rho}_{RA} | (\mathrm{id}_R \otimes \mathcal{N}_{A\to A})(\psi^{\rho}_{RA}) | \psi^{\rho}_{RA} \rangle,$$

Howard Barnum. Quantum Rate-Distortion Coding. Phys. Rev. A 62, 042309 (2000)

More General: Distortion Observable

Distortion based on entanglement fidelity is sensible, but having a **distortion observable** leads to a more general framework

Let Δ_{RB} be a **distortion observable** on the systems *RB*. Then define distortion *d* that a channel causes to a state ρ to be

$$d(\rho, \mathcal{N}) \equiv \mathrm{Tr}\left(\Delta_{RB}\left((\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(\psi^{\rho}_{RA})\right)\right)$$

The average distortion for a block of length *n* is then

$$\overline{d}(\rho, \mathcal{D}_n \circ \mathcal{E}_n) \equiv \frac{1}{n}\sum_{i=1}^{n} d(\rho, \mathcal{F}_n^{(i)})$$

$$= \mathrm{Tr}\left(\overline{\Delta}\left((\mathrm{id}_R \otimes \mathcal{F})\left((\psi^{\rho}_{RA})^{\otimes n}\right)\right)\right).$$

where $\overline{\Delta} \equiv \frac{1}{n}\sum_{i=1}^{n} I^{\otimes(i-1)} \otimes \Delta \otimes I^{\otimes(n-i)}$

Winter and Ahlswede. Quantum Rate Distortion Theory. Unpublished manuscript (2001)

Best Known Characterization of QRD

The best known characterization of the **quantum rate distortion function** is in terms of a regularized entanglement of purification:

$$R^q(D) = \lim_{k\to\infty} \frac{1}{k}\left[\min_{\mathcal{N}^{(k)}\ :\ \overline{d}(\rho^{\otimes k},\mathcal{N}^{(k)})\leq D} E_p\left(\rho^{\otimes k},\mathcal{N}^{(k)}\right)\right]$$

where $E_p(\rho,\mathcal{N}) = E_p((\mathrm{id}_R \otimes \mathcal{N})(\psi^{\rho}_{RA}))$

$$E_p(\omega_{AB}) = \min_{\mathcal{N}} H\left((\mathrm{id}_B \otimes \mathcal{N}_{E\to E'})(\psi^{\omega}_{BE})\right)$$

Disadvantages of this characterization:

Regularized and EoP is difficult to compute

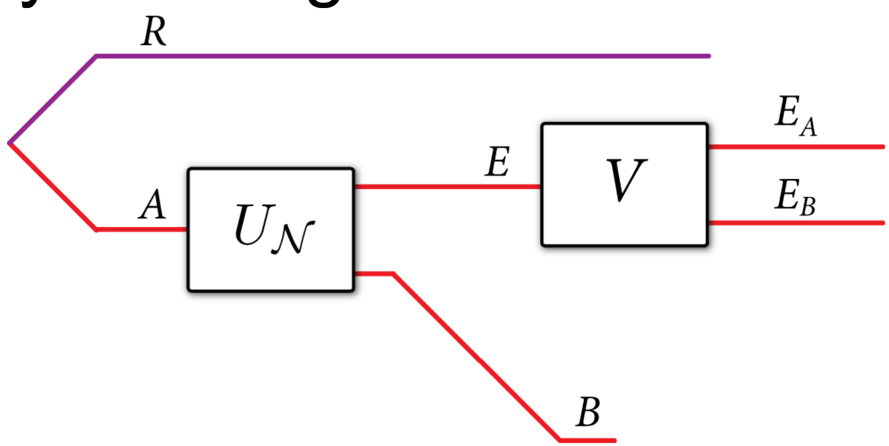
Datta, Hsieh, and Wilde. IEEE Trans. Inf. Theory vol. 59, no. 1, pp. 615-630 (January 2013)

How to Achieve EoP Rate

Consider single-letter quantity: $\min_{\mathcal{N}\ :\ \overline{d}(\rho,\mathcal{N})\leq D} E_p(\rho,\mathcal{N})$

Compute the channel that minimizes the EoP while meeting distortion constraint

On each copy of the source, Alice simulates an isometric extension *U* of this channel, followed by some isometry *V* acting on the environment:



Alice can then Schumacher compress systems *B* and E_B at a rate $H(B\,E_B)$.

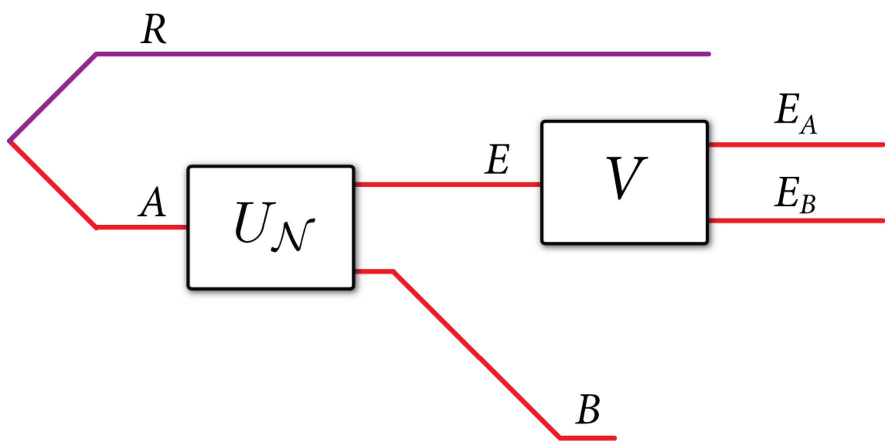
By choosing *V* optimally, this achieves the EoP rate.

Double-block this strategy to get regularization!

Distinction between Classical and Quantum Settings

The former strategy is simple, and it is quantum effects that allow for this to be optimal in the regularized limit.

Suppose classical systems in the former strategy and that U and V are stochastic maps:



Then $H(B\,E_B) \geq H(B)$ so that there is no point in doing V.
But quantumly, we can have that $H(B\,E_B) < H(B)$ (due to entanglement) so that this strategy becomes optimal in the regularized limit.

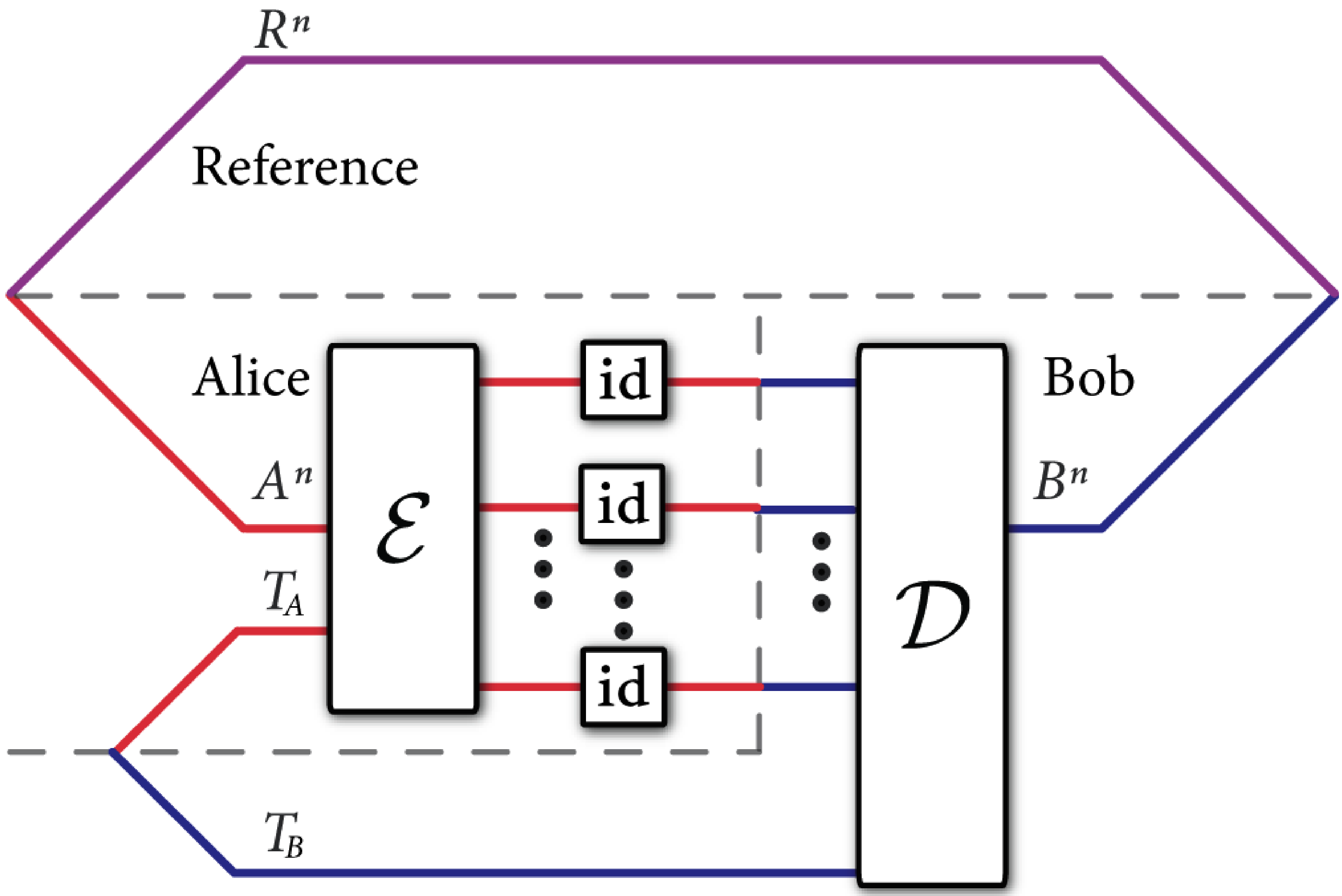
Bounding the QRD

Due to the aforementioned difficulties, we could also try to obtain bounds on the quantum rate distortion function by assuming that

1) Shared entanglement is for free

2) Classical communication is for free (like Devetak-Berger)

Entanglement-Assisted QRD



Distortion is defined similarly as in the unassisted setting.

The EA QRD provides a lower bound to the unassisted QRD.

Datta, Hsieh, and Wilde. IEEE Trans. Inf. Theory vol. 59, no. 1, pp. 615-630 (January 2013)

EA Quantum Rate Distortion Function

EA QRD Theorem:

$$R^q_{ea}(D) = \frac{1}{2}\left[\min_{\mathcal{N}\ :\ d(\rho,\mathcal{N})\leq D} I\left(R;B\right)_\omega\right]$$

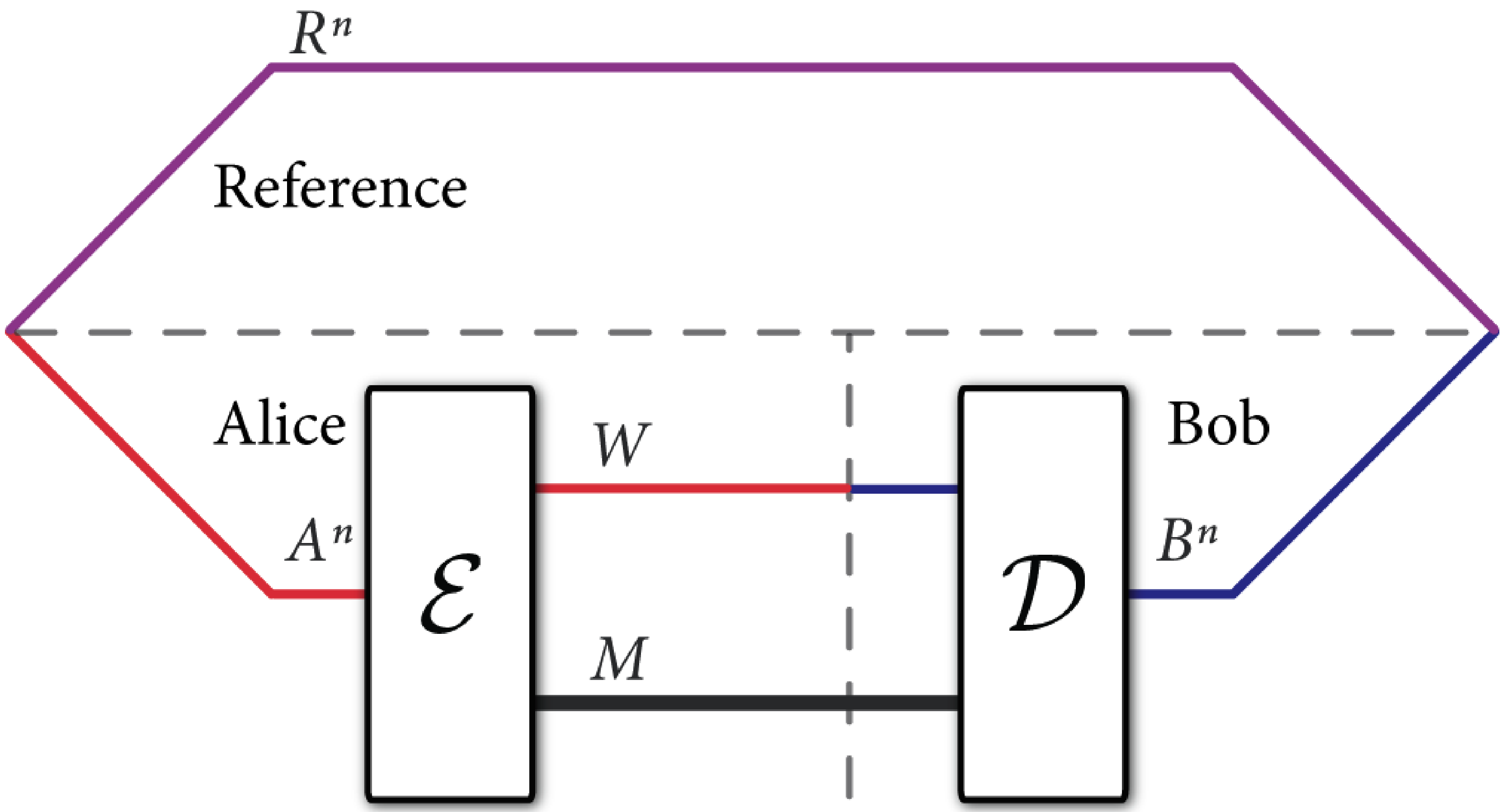
where $\omega_{RB} = \mathcal{N}_{A\to B}(\psi^\rho_{RA})$

Achievability: Compute the channel meeting the minimum (easy due to convexity) and use EA quantum reverse Shannon theorem.

Converse: Consider most general protocol with EA, some entropic manipulations, and then follow standard converse proof in Cover and Thomas

Datta, Hsieh, and Wilde. IEEE Trans. Inf. Theory vol. 59, no. 1, pp. 615-630 (January 2013)

Classically-Assisted QRD



The QRD task assisted by a classical side channel.

Provides another lower bound to the unassisted QRD function.

Devetak and Berger. IEEE Trans. Inf. Theory, 48(6):1580-1589, June 2002.
Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

Classically-Assisted QRD Function

CA QRD Theorem:

$$R^{q}_{\rightarrow}(D) = \lim_{k\rightarrow\infty}\frac{1}{k}\min_{\substack{\mathcal{N}^{(k)}:\\ \overline{d}(\rho^{\otimes k},\mathcal{N}^{(k)})\leq D}}\left[E_F(\rho^{\otimes k},\mathcal{N}^{(k)})\right]$$

where $E_F(\rho,\mathcal{N}) = E_F((\mathrm{id}_R\otimes\mathcal{N})(\psi^{\rho}_{RA}))$

$$E_F(\omega_{AB}) \equiv \min_{\{p(x),|\psi^x_{AB}\rangle\}:\omega=\sum_x p(x)\psi^x}\sum_x p(x)\,H(A)_{\psi^x},$$

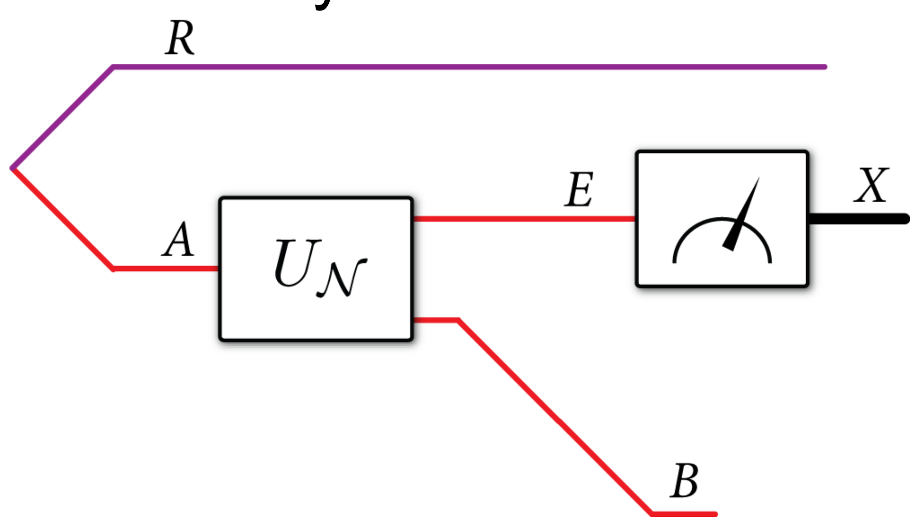
Devetak and Berger. IEEE Trans. Inf. Theory, 48(6):1580-1589, June 2002.
Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

How to Achieve EoF Rate

Consider single-letter quantity: $\min_{\mathcal{N} \,:\, \overline{d}(\rho,\mathcal{N}) \leq D} E_F(\rho, \mathcal{N})$

Compute the channel that minimizes the EoF while meeting distortion constraint

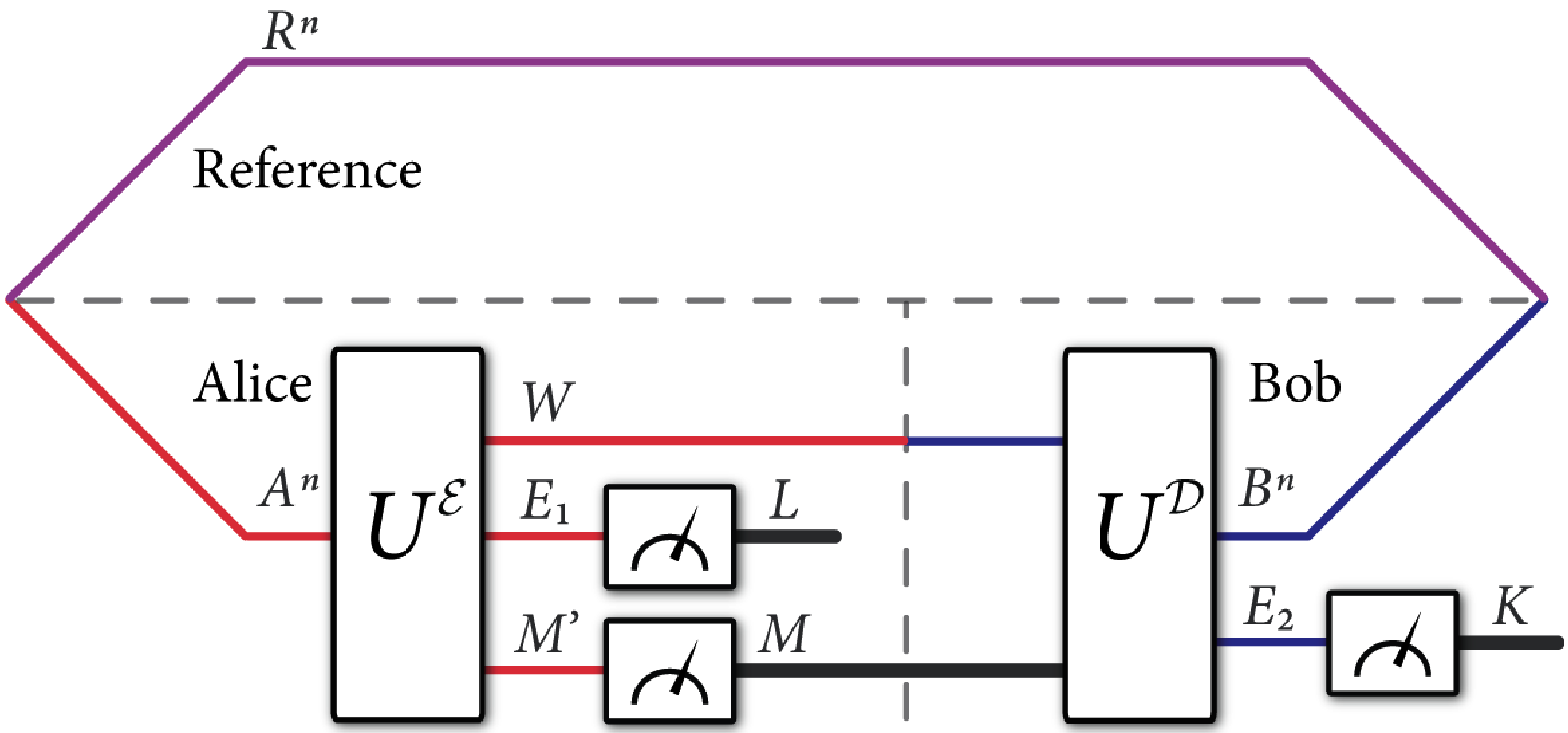
On each copy of the source, Alice simulates an isometric extension *U* of this channel, followed by a measurement of the environment:



Double-block this strategy to get regularization!

Alice can then Schumacher compress system *B* conditional on sequence x^n at a rate $H(B|X)$, while sending x^n to Bob. Choosing the measurement optimally gives the EoF rate.

Converse: Classically-Assisted QRD



Simulate general protocol by the above one.

$$\begin{aligned}\log(\dim \mathcal{H}_W) &\geq H(W)_\omega \geq H(W|LM)_\omega = H(R^n|LM)_\omega \\ &\geq H(R^n|LMK)_\sigma \geq E_F(\sigma_{R^nB^n}) \\ &\geq \min_{\substack{\mathcal{N}^{(n)}: \\ d(\rho^{\otimes n}, \mathcal{N}^{(n)}) \leq D}} \left[E_F(\rho^{\otimes n}, \mathcal{N}^{(n)}) \right].\end{aligned}$$

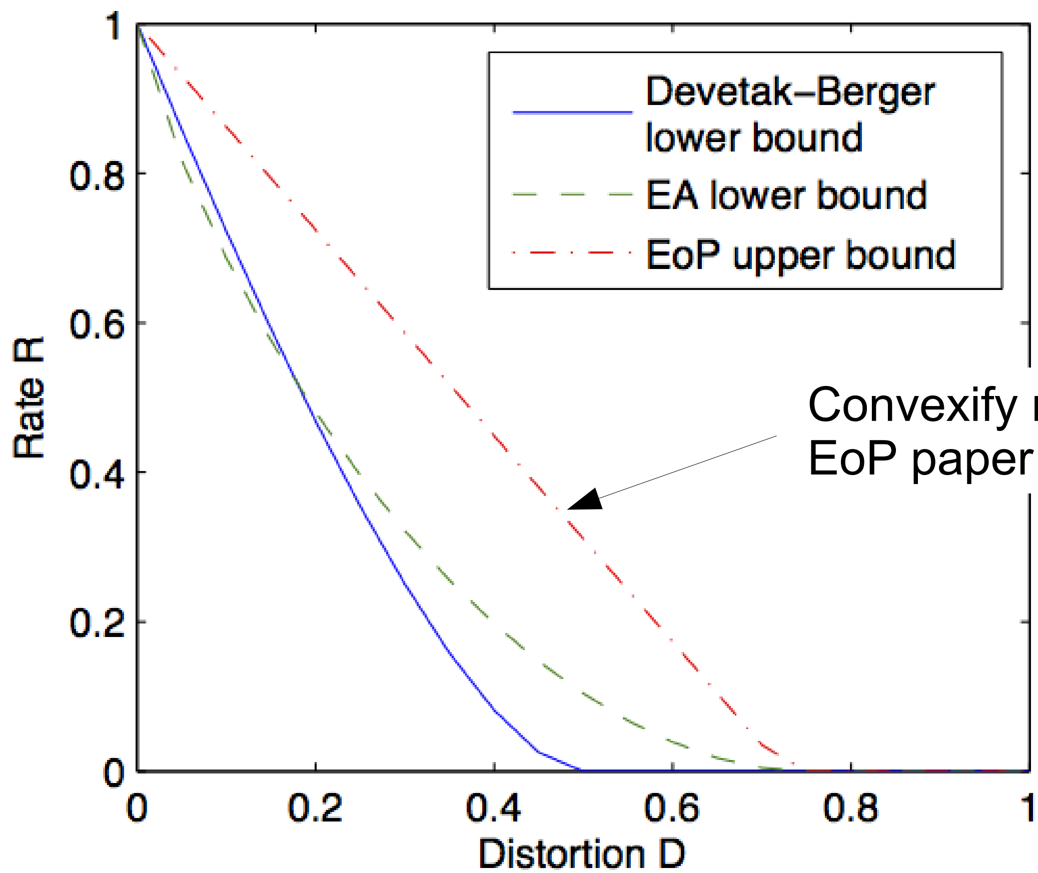
Simplest Example: Isotropic Qubit Source

Quantum Info. Source — maximally mixed qubit state

(*quantum analog of simplest classical example*)

Combine all of the previous bounds:

EoP upper bound, EA lower bound, and CA lower bound



Convexify numerical results of EoP paper arXiv:quant-ph/0202044

Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

Simplest Example: Isotropic Qubit Source

Single-letter expression for EA QRD:

$$R_{ea}^{q}(D) = \begin{cases} 1 - \frac{1}{2} H\left(\left\{1 - D, \frac{D}{3}, \frac{D}{3}, \frac{D}{3}\right\}\right) & \text{if } 0 \le D \le \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \le D \le 1, \end{cases}$$

Single-letter expression for CA QRD:

$$R_{\rightarrow}^{q}(D) = \begin{cases} h_2\left(\frac{1}{2} + \sqrt{D(1-D)}\right) & : 0 \le D < \frac{1}{2} \\ 0 & : \frac{1}{2} \le D \le 1 \end{cases}$$

Isotropic qubit source has plenty of symmetry:

Take optimal channel, can prepend and append any Clifford unitary and distortion and information measure remain invariant.
Clifford twirled channel has same distortion but info. measures only go down (from their convexity)
This implies optimal channel is a depolarizing channel.

QRD with Quantum Side Info.

Wyner and Div proved a theorem regarding rate distortion coding with quantum side info. (QSI)

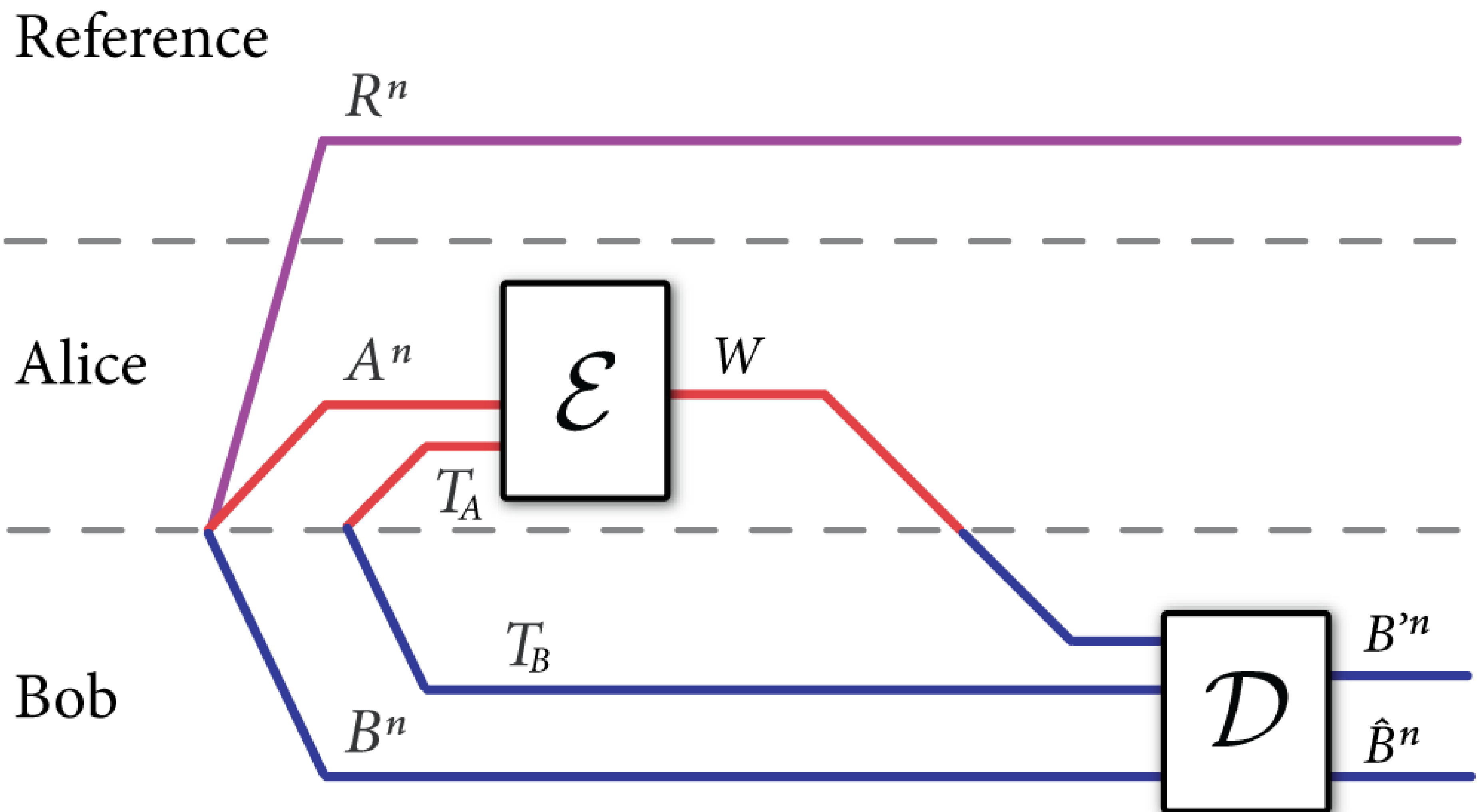
We would like to prove some quantum analog of this theorem to understand how QSI can help

Simplify by assuming **entanglement assistance**

Another Assumption: The protocol can cause only a negligible disturbance to the QSI

Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

QRD with Quantum Side Info.



Distortion is with respect to the *A* systems

Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

EA-QRD-QSI Theorem

Theorem:

$$R^q_{ea,qsi}(D) = \frac{1}{2}\left[\min_{\mathcal{N}\ :\ d(\rho,\mathcal{N})\leq D} I(R;B'|B)_\sigma\right]$$

where $\sigma_{RB'B} \equiv \mathcal{N}_{A\to B'}(\phi^\rho_{RAB})$

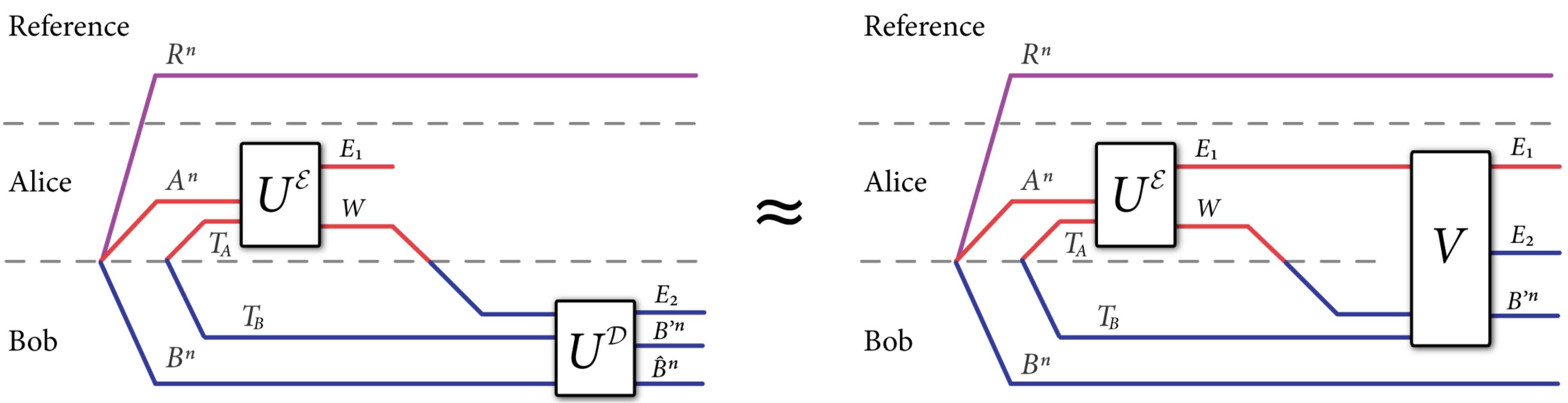
Advantage: Single-letter expression

Achievability: Use **quantum state redistribution**.
Alice simulates channel meeting distortion constraint on every *A* system. This does not disturb QSI in the limit, and rate of q. comm. is as given above.

Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

Converse for EA-QRD-QSI Theorem

Converse: Consider most general protocol for this task and purify everything...



Invoke assumption that QSI cannot be disturbed too much to infer the existence of a "nearby decoder" not acting on QSI (*Uhlmann's theorem*)...

Datta, Wilde, Hsieh, and Winter. arXiv:1212.5316

Conclusion and Current Work

Characterization of QRD in terms of regularized entanglement of purification

Lower bounds on QRD from auxiliary resources such as classical communication and entanglement

QRD with quantum side information simple when EA is available.

More complicated without EA (*discussed in paper*)

Most important open question: Find a better characterization of QRD function