Finite blocklength converse bounds for quantum channels (arXiv:1210.4722)

Will Matthews (University of Cambridge)
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Classical data over quantum channels

- Upper bounds on rate for classical communication over quantum channels at finite blocklength n and error probability ϵ for
  - Entanglement assisted (EA) (i.i.d. asymptotics: BSST formula).
  - Unassisted codes (i.i.d. asymptotics: Regularized Holevo bound).

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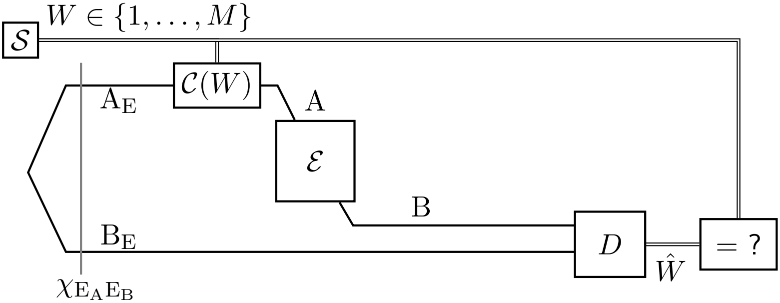
- Relates coding to hypothesis testing in a generalisation of the classical converse of Polyanskiy–Poor–Verdú, and quantum converse of Wang-Renner.
- Converse for EA codes has some advantages over the converse of Datta and Hsieh (arXiv:1105.3321).

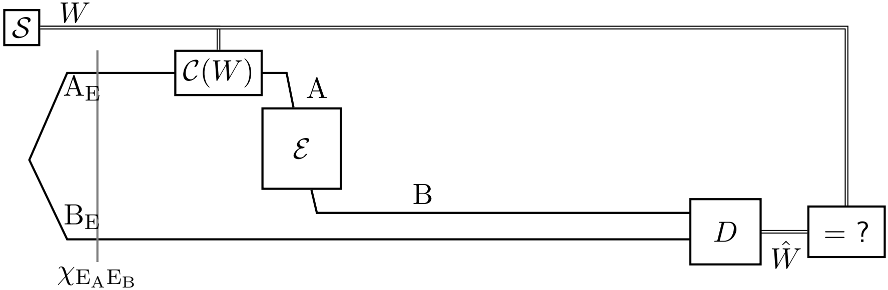
Codes for quantum channels



Avg. input: $\rho_{\mathrm{A}} = \frac{1}{M}\sum_w \mathcal{C}(w)_{\mathrm{A|E_A}}[\chi_{\mathrm{E_A}}]$
$M^{\mathrm{E}}_{\epsilon}(\mathcal{E},\rho)$: Max. M for EA code with avg. input ρ and error probability ϵ over $\mathcal{E}$.

$$\begin{aligned} 1-\epsilon &= \Pr(W=\hat{W}|\text{code},\mathcal{E},\mathcal{S}) \\ &= \frac{1}{M}\sum_w \mathrm{Tr} D(w)_{\mathrm{BE_B}}\mathcal{E}_{\mathrm{B|A}}[C(w)_{\mathrm{A|E_A}}[\chi_{\mathrm{E_A E_B}}]]. \end{aligned}$$

Codes to tests



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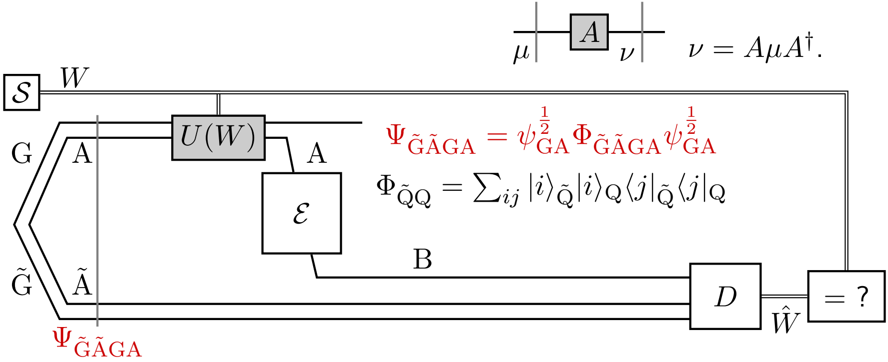
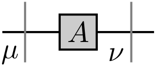
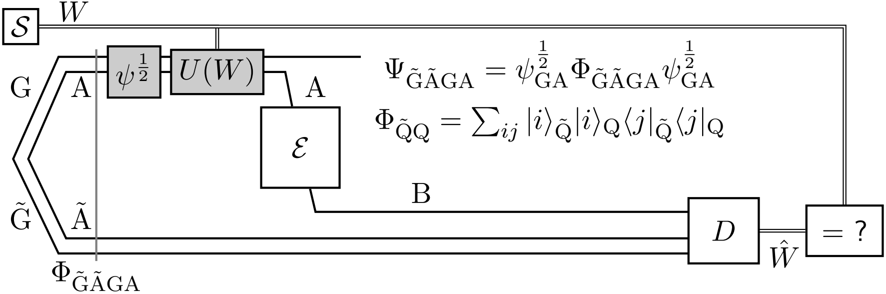


Codes to tests

$$\rho_{\mathrm{A}} = \frac{1}{M}\sum_w \mathrm{Tr}_{\mathrm{G}} U(w)\psi_{\mathrm{AG}} U(w)^\dagger$$



$$\nu = A\mu A^\dagger.$$



$$\Psi_{\tilde{\mathrm{G}}\tilde{\mathrm{A}}\mathrm{GA}} = \psi_{\mathrm{GA}}^{\frac{1}{2}} \Phi_{\tilde{\mathrm{G}}\tilde{\mathrm{A}}\mathrm{GA}} \psi_{\mathrm{GA}}^{\frac{1}{2}}$$

$$\Phi_{\tilde{\mathrm{Q}}\mathrm{Q}} = \sum_{ij} |i\rangle_{\tilde{\mathrm{Q}}} |i\rangle_{\mathrm{Q}} \langle j|_{\tilde{\mathrm{Q}}} \langle j|_{\mathrm{Q}}$$

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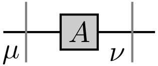

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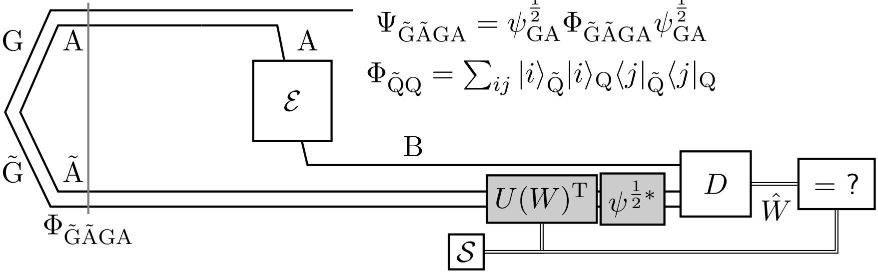

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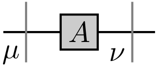

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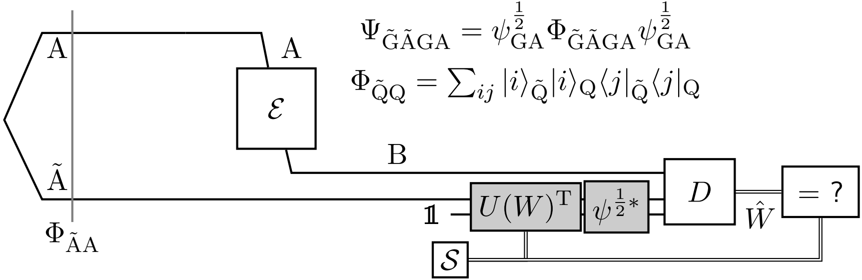

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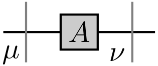

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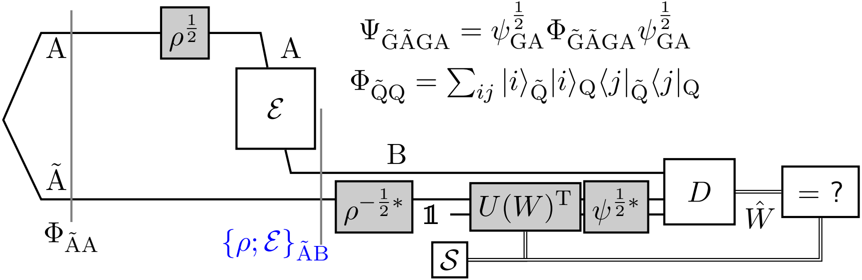

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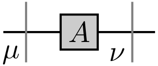

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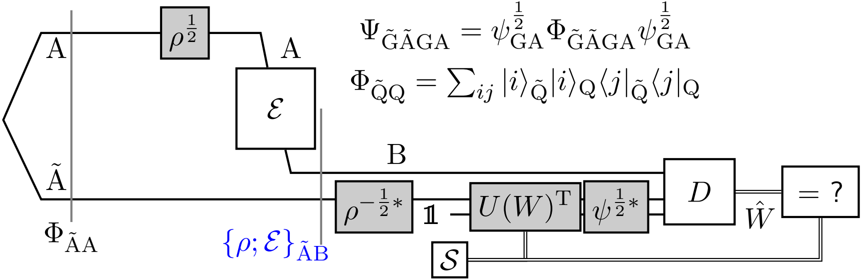

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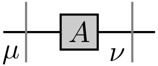

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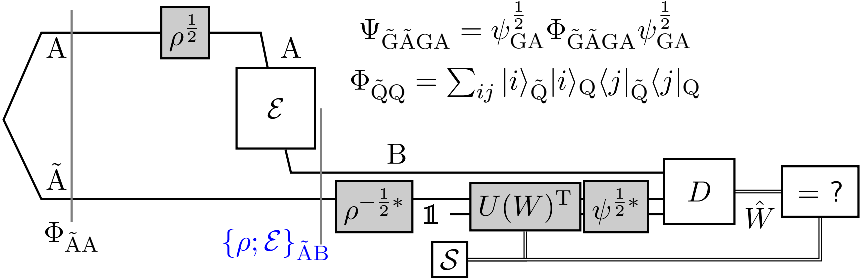

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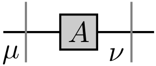

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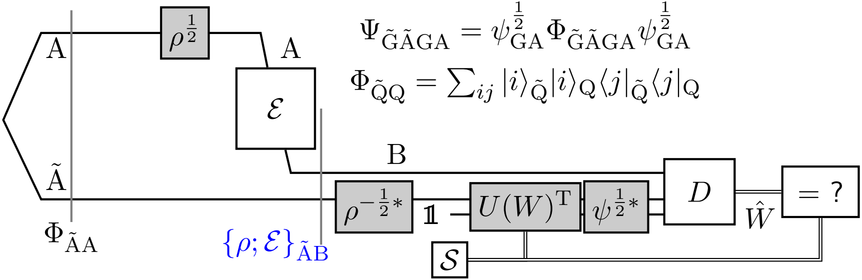

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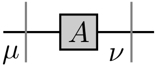

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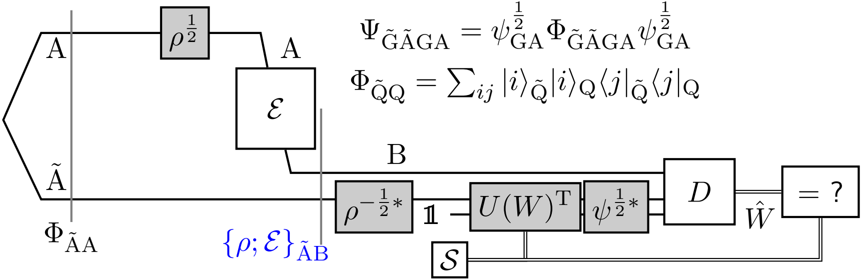

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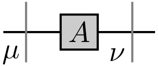

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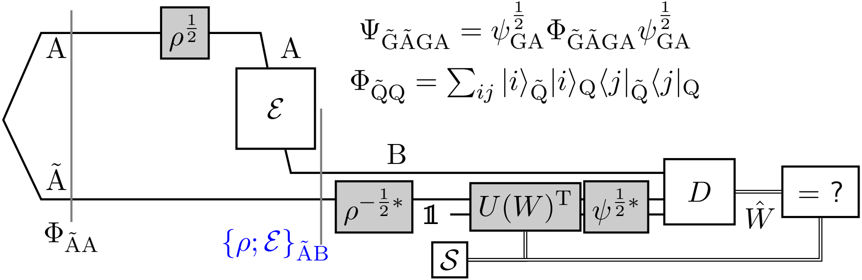

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Meta-converse for quantum channels

- Generalises classical meta-converse (for the case of finite alphabets)[1].

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- Furthermore: **Unassisted** codes maps to **Local** Tests.

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$\beta_\epsilon^{\mathbf{\Omega}}(\tau_0, \tau_1) := \min \beta(T) (= \min \mathrm{Tr} T \tau_1)$ such that $\alpha(T) = 1 - \mathrm{Tr} T \tau_0 \le \epsilon$ and $T \in \mathbf{\Omega}$.

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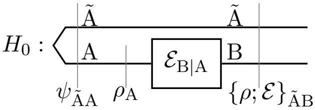
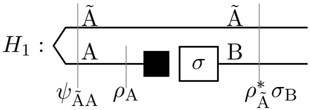
Classes on bipartite systems: $\mathbf{L}$ is local tests; $\mathbf{LC1}$ is one-way classical comm. from A to B, $\mathbf{PPT} : 0 \leq \Gamma_{\mathrm{B}}[T_{\tilde{\mathrm{A}}\mathrm{B}}] \leq \mathbb{1}$;
$\mathbf{\Omega}$ superscript omitted = any test.

The main bounds





Entanglement-assisted codes with avg. input ρ:

$$M_\epsilon^{\mathrm{E}}(\mathcal{E},\rho) \leq \left(\max_\sigma \beta_\epsilon(\{\rho;\mathcal{E}\}_{\tilde{A}B}, \rho^*_{\tilde{A}}\sigma_B)\right)^{-1}$$

Unassisted codes with avg. input ρ:

$$M_\epsilon(\mathcal{E},\rho) \leq \left(\max_\sigma \beta_\epsilon^{\mathbf{L}}(\{\rho;\mathcal{E}\}_{\tilde{A}B}, \rho^*_{\tilde{A}}\sigma_B)\right)^{-1}$$

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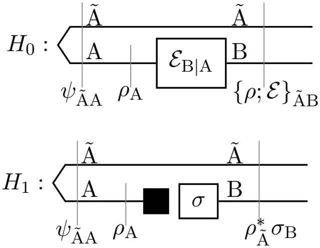

Entanglement-assisted codes:

$$M_\epsilon^{\mathrm{E}}(\mathcal{E}) \leq \left(\min_\rho \max_\sigma \beta_\epsilon(\{\rho; \mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}, \rho^*_{\tilde{\mathrm{A}}} \sigma_{\mathrm{B}}) \right)^{-1}$$

Unassisted codes:

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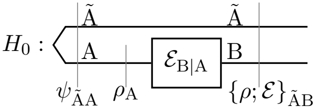

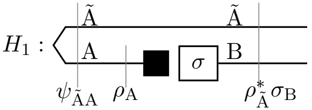

Entanglement-assisted codes:

$$M^{\mathrm{E}}_{\epsilon}(\mathcal{E}) \leq \left(\min_{\rho}\max_{\sigma}\beta_{\epsilon}(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}, \rho^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}})\right)^{-1}$$

Unassisted codes:

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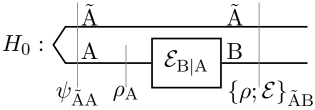

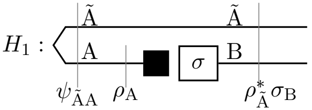

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Unassisted codes:

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Properties of the bounds

See Y. Polyanskiy's study of classical bound[2].

- $\beta_\epsilon^{\mathbf{\Omega}}(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}, \rho^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}})$ is
  - Concave in σ.
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- Asymptotically tight (for $\mathbf{\Omega} = \mathbf{ALL}, \mathbf{LC1}, \mathbf{L}$)

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Concavity in σ

$$\begin{aligned}\beta_\epsilon(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}, \rho^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}) = &\min \mathrm{Tr} T_{\tilde{\mathrm{A}}\mathrm{B}} \rho^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}} \\ &\mathrm{Tr} T_{\tilde{\mathrm{A}}\mathrm{B}}\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \geq 1-\epsilon \\ &T \in \mathbf{\Omega}\end{aligned}$$

Minimum over concave (linear, in fact) functions of σ is concave.

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$$=\lambda_i\rho(i)^{\frac{1}{2}*}_{\tilde{\mathrm{A}}}\mathcal{E}_{\mathrm{B}|\mathrm{A}}[\Phi_{\tilde{\mathrm{A}}\mathrm{A}}]\rho(i)^{\frac{1}{2}*}_{\tilde{\mathrm{A}}}M(i)^{\dagger}_{\tilde{\mathrm{A}}} = \lambda_i\{\rho_i;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}.$$

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$$M(i)_{\tilde{A}}\{\rho;\mathcal{E}\}_{\tilde{A}B}M(i)^\dagger_{\tilde{A}} = M(i)_{\tilde{A}}\rho_{\tilde{A}}^{\frac{1}{2}*}\mathcal{E}_{B|A}[\Phi_{\tilde{A}A}]\rho_{\tilde{A}}^{\frac{1}{2}*}M(i)^\dagger_{\tilde{A}}$$
$$=\lambda_i\rho(i)_{\tilde{A}}^{\frac{1}{2}*}\mathcal{E}_{B|A}[\Phi_{\tilde{A}A}]\rho(i)_{\tilde{A}}^{\frac{1}{2}*}M(i)^\dagger_{\tilde{A}} = \lambda_i\{\rho_i;\mathcal{E}\}_{\tilde{A}B}.$$

Similarly, $M(i)_{\tilde{A}}\rho^*_{\tilde{A}}\sigma_B M(i)^\dagger_{\tilde{A}} = \lambda_i\rho(i)^*_{\tilde{A}}\sigma_B$.

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Similarly, $M(i)_{\tilde{\mathrm{A}}}\rho^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}M(i)^\dagger_{\tilde{\mathrm{A}}} = \lambda_i\rho(i)^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}$. With probability λ_i, outcome i occurs. Alice sends i to Bob and they use an optimal strategy in $\mathbf{\Omega}$ with $\alpha = \epsilon$ to test for $\{\rho(i);\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}$ against $\rho(i)^*_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}$.

Convexity in ρ

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$\tilde{\mathrm{A}}$lice performs measurement with meas. op.
$M(i)_{\tilde{\mathrm{A}}} = \sqrt{\lambda_i}\rho(i)^{\frac{1}{2}*}\rho^{\frac{1}{2}*}$ for outcome i.

$$M(i)_{\tilde{\mathrm{A}}}\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}M(i)^{\dagger}_{\tilde{\mathrm{A}}} = M(i)_{\tilde{\mathrm{A}}}\rho^{\frac{1}{2}*}_{\tilde{\mathrm{A}}}\mathcal{E}_{\mathrm{B}|\mathrm{A}}[\Phi_{\tilde{\mathrm{A}}\mathrm{A}}]\rho^{\frac{1}{2}*}_{\tilde{\mathrm{A}}}M(i)^{\dagger}_{\tilde{\mathrm{A}}}$$
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Similarly, $M(i)_{\tilde{\mathrm{A}}}\rho^{*}_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}M(i)^{\dagger}_{\tilde{\mathrm{A}}} = \lambda_i\rho(i)^{*}_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}$. With probability λ_i, outcome i occurs. Alice sends i to Bob and they use an optimal strategy in $\mathbf{\Omega}$ with $\alpha = \epsilon$ to test for $\{\rho(i);\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}$ against $\rho(i)^{*}_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}}$. This test has $\alpha = \epsilon$ and $\beta = \sum_i \lambda_i\beta^{\mathbf{\Omega}}_{\epsilon}(\{\rho(i);\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}, \rho(i)^{*}_{\tilde{\mathrm{A}}}\sigma_{\mathrm{B}})$.

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Using symmetry

Suppose that there is a group G with an action g_{A} ($g_{\tilde{\mathrm{A}}}$) on states of A (and of $\tilde{\mathrm{A}}$) given by

$$g_{\mathrm{A}}[\tau_{\mathrm{A}}] = U(g)_{\mathrm{A}} \tau_{\mathrm{A}} U(g)_{\mathrm{A}}^{\dagger},$$

and an action g_{B} on states of B

$$g_{\mathrm{B}}[\tau_{\mathrm{B}}] = V(g)_{\mathrm{B}} \tau_{\mathrm{B}} V(g)_{\mathrm{B}}^{\dagger},$$

where U and V are unitary representations of G, such that

$$\forall g \in G: \; \mathcal{E}_{\mathrm{B}|\mathrm{A}}(g_{\mathrm{A}}[\rho_{\mathrm{A}}]) = g_{\mathrm{B}}[\mathcal{E}_{\mathrm{B}|\mathrm{A}}(\rho_{\mathrm{A}})].$$

We call such a channel G-covariant.

Using symmetry

For any $\Omega \supseteq \mathbf{L}$ and for all $g \in G$,

$$\beta_\epsilon^{\mathbf{\Omega}}(\{g_{\mathrm{A}}[\rho];\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| g^*_{\tilde{\mathrm{A}}}[\rho^*_{\tilde{\mathrm{A}}}] g_{\mathrm{B}}[\sigma_{\mathrm{B}}]) = \beta_\epsilon^{\mathbf{\Omega}}(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| \rho^*_{\tilde{\mathrm{A}}} \sigma_{\mathrm{B}}).$$

Therefore

$$\begin{aligned}
\max_{\sigma_{\mathrm{B}}} \beta_\epsilon^{\mathbf{\Omega}}(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| \rho^*_{\tilde{\mathrm{A}}} \sigma_{\mathrm{B}}) =& \beta_\epsilon^{\mathbf{\Omega}}(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| \rho^*_{\tilde{\mathrm{A}}} \sigma^0_{\mathrm{B}}) \\
=& \beta_\epsilon^{\mathbf{\Omega}}(\{g_{\mathrm{A}}[\rho];\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| g^*_{\tilde{\mathrm{A}}}[\rho^*_{\tilde{\mathrm{A}}}] g_{\mathrm{B}}[\sigma^0_{\mathrm{B}}]) \\
\leq& \max_{\sigma_{\mathrm{B}}} \beta_\epsilon^{\mathbf{\Omega}}(\{g_{\mathrm{A}}[\rho];\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| g^*_{\tilde{\mathrm{A}}}[\rho^*_{\tilde{\mathrm{A}}}] \sigma_{\mathrm{B}})
\end{aligned}$$

and since g has an inverse in G

$$\max_{\sigma_{\mathrm{B}}} \beta_\epsilon^{\mathbf{\Omega}}(\{\rho;\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| \rho_{\tilde{\mathrm{A}}} \sigma_{\mathrm{B}}) = \max_{\sigma_{\mathrm{B}}} \beta_\epsilon^{\mathbf{\Omega}}(\{g_{\mathrm{A}}[\rho];\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}} \| g^*_{\tilde{\mathrm{A}}}[\rho^*_{\tilde{\mathrm{A}}}] \sigma_{\mathrm{B}})$$

Using symmetry

Let $\bar{\rho}_\mathrm{A} := \int_G g_\mathrm{A}[\rho_\mathrm{A}]\mu(g)$ where μ is Haar measure on G. Since $\max_{\sigma_\mathrm{B}} \beta_\epsilon^{\mathbf{\Omega}}(\{\bar{\rho};\mathcal{E}\}_{\tilde{\mathrm{A}}\mathrm{B}}\|\bar{\rho}_{\tilde{\mathrm{A}}}\sigma_\mathrm{B})$ is convex in ρ, by Jensen's inequality and preceding result

$$\begin{aligned}
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\end{aligned}
$$

So, for G-covariant $\mathcal{E}$ we can restrict to G-invariant ρ in the optimization for the bounds (when $\mathbf{\Omega} \supseteq \mathbf{LC1}$).
Similarly we can restrict to G-invariant σ in all the bounds.

Example: EA coding over Depolarising Channel

- $\mathcal{D}_{\mathrm{B|A}}[\tau_{\mathrm{A}}] = (1-p)\tau_{\mathrm{B}} + p\mathrm{Tr}(\tau_{\mathrm{B}})\mu_{\mathrm{B}}$ where $\mu \propto \mathbb{1}$ is the maximally mixed state.

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- $M_{\epsilon}^{\mathrm{E}}(\mathcal{D}^{\otimes n}) \leq \beta_{\epsilon}(\phi(p)_{\tilde{\mathrm{A}}^n\mathrm{B}^n}^{\otimes n} \| \mu_{\tilde{\mathrm{A}}^n}\mu_{\mathrm{B}^n})^{-1}$

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- n uses has the covariance group $S_n \ltimes \mathrm{U}(d)^{\times n}$
- U and V are irreducible so maximally mixed ρ and σ are optimal. $\mathcal{D}_{\mathrm{B|A}}[\phi_{\tilde{\mathrm{A}}\mathrm{A}}] = \phi(p) := (1-p)\phi_{\tilde{\mathrm{A}}\mathrm{B}} + p\mu_{\tilde{\mathrm{A}}}\mu_{\mathrm{B}}$ is an isotropic state.
- $M_{\epsilon}^{\mathrm{E}}(\mathcal{D}^{\otimes n}) \leq \beta_{\epsilon}(\phi(p)_{\tilde{\mathrm{A}}^n\mathrm{B}^n}^{\otimes n} \| \mu_{\tilde{\mathrm{A}}^n}\mu_{\mathrm{B}^n})^{-1}$
- Since the arguments commute, and there are only two distinct eigenvalues (eigenprojectors ϕ and $\mathbb{1} - \phi$) $\beta_{\epsilon}(\phi(p)_{\tilde{\mathrm{A}}^n\mathrm{B}^n}^{\otimes n} \| \mu_{\tilde{\mathrm{A}}^n}\mu_{\mathrm{B}^n}) = \beta_{\epsilon}((\mu, 1-\mu)^{\otimes n}, (\lambda, 1-\lambda)^{\otimes n})$ where $\mu = (1-p) + p/d^2$ and $\lambda = 1/d^2$. This is simple to compute.

Example: EA coding over Depolarising Channel

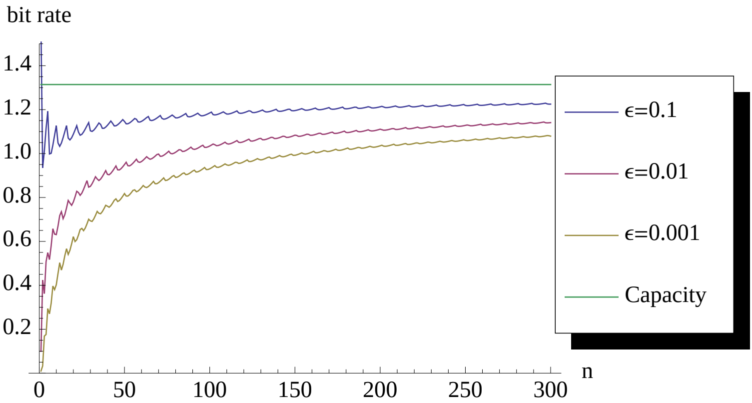

Figure: The upper bound on the rate for entanglement assisted codes over the p=0.15 depolarising channel for three different error probabilities.

Outlook

- Easy solutions for other simple channels.
- Relationship to Datta-Hsieh bound?
- Second order asymptotics for EA codes (converse part of Strassen-like result)?
- Infinite dimensional Hilbert spaces?