# Exponential Decay of Correlations Implies Area Law: an i.i.d. story

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# Plan

**I. Preliminaria**

- o Area law
  - Definition
  - a (tiny) bit of history/motivation
- o Exponential Decay of Correlations
  - Definition/examples

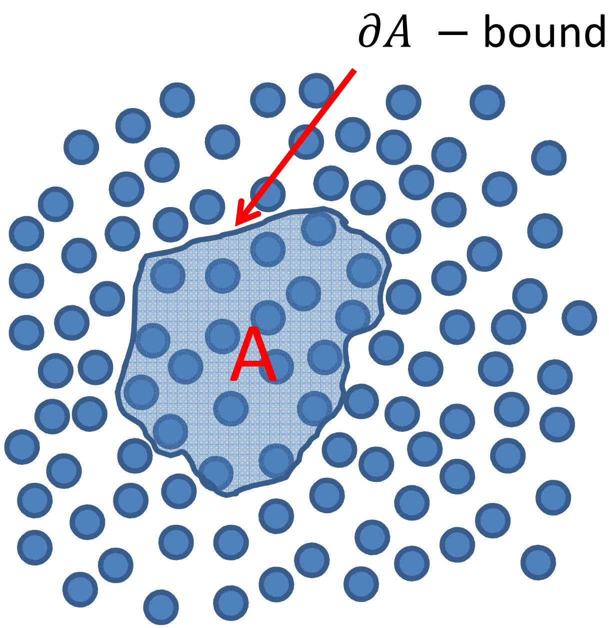
**II. Area Law versus EDC – what was known before**

- o Why EDC might imply by Area Law ?
- o Why not really?
  - Quantum data hiding states
- o Why yet it still might?
  - Random states

**III. Main result**

- Statement
- Sketch of proof, with massive cheating

# Area Law

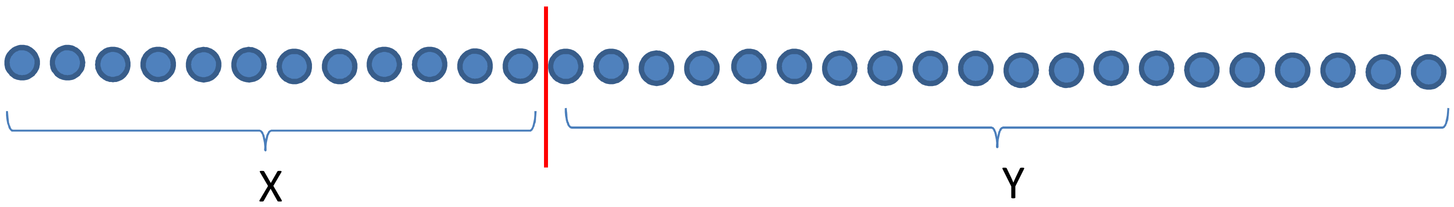



We say that a state $\rho$ satisfies Area Law if

$$S(A) \sim \partial(A)$$

**Area Law in 1D:** A state $\rho_{1 \dots n}$ satisfies Area Law if for all cuts $(X, Y)$ of the chain with $X = [1, \dots, k]$, $Y = [k+1, \dots, n]$

$$S(\rho_X) \leq const$$



# Area Law: a bit of history

- Thermodynamics of black hole: the entropy should scale as an area (Bekenstein, Hawking)
  → quantum field theories: Średnicki 1993

- Quantum Information – Audenaert, Eisert, Plenio, Werner '02
  → entanglement of a ground state should scale as the size of the boundary

- Complexity of description of states of quantum many-body systems:
  – Fannes, Nachtergaele, Werner, 1992
  – Vidal 2003

**A generic state of n qubits is complex:** It requires exponential number of parameters to describe:

$$|\psi\rangle_{1,\dots,n} = \sum_{i_1,\dots,i_n} a_{i_1} \dots a_{i_n} |i_1\rangle \dots |i_n\rangle$$

States satisfying **Area Law** admit **simple description** – can be efficiently simulated by classical computer. (Vidal 2003, can be also found in Fannes et al.)

# States that satisfy Area Law

Rigorous resutls

(Aharonov *et al* '07; Irani '09, Irani, Gottesman '09)
Ground states 1D Ham. with *volume* law **$S(X) \geq \Omega(vol(X))$**
Connection to QMA-hardness

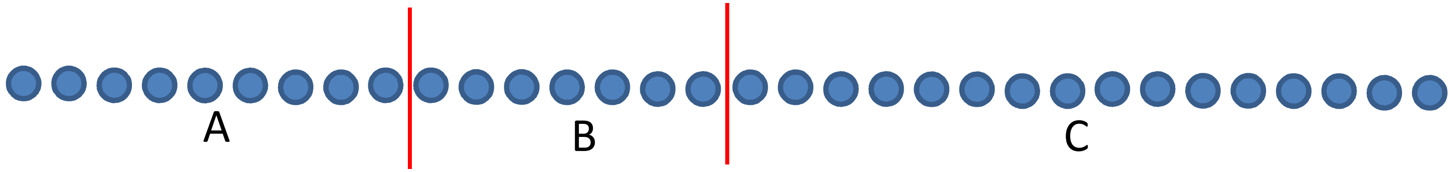
(Hastings '07)
Ground states 1D *gapped* local Ham. **$S(X) \leq 2^{O(1/\Delta)}$**
Analytical Proof: Lieb-Robinson bounds, etc…

(Wolf, Verstraete, Hastings, Cirac '07)
Thermal states of local Ham. **$I(X:Y) \leq O(Area(X)/\beta)$**
Proof from Jaynes' principle

(Arad, Kitaev, Landau, Vazirani '12) **$S(X) \leq O(1/\Delta)$**
Ground states 1D *gapped* local Ham.
Combinatorial Proof: Chebyshev polynomials, etc…

# Exponential Decay of correlations

Correlations: $$Cor(X{:}Y) = \max_{\|M\|\le 1, \|N\|\le 1} \left| tr\left(M \otimes N(\rho_{XY} - \rho_X \otimes \rho_Y)\right)\right|$$



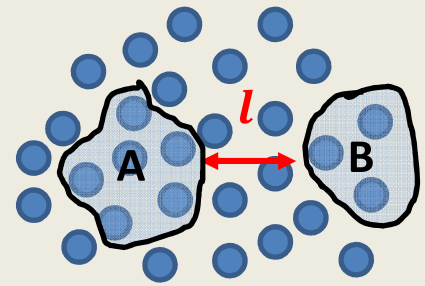

We say that the state $\rho_{1\ldots n}$ has **Exponential Decay of Correlations** if there exist $(\boldsymbol{l_0}, \boldsymbol{\xi})$ such that for any subsystems separated by $l \ge l_0$ sites we have

$$Cor(A{:}C) \le 2^{-l/\xi}$$

$\xi$ - correlation length (range)
$l_0$ - minimum distance the correlations start decaying

**Example 1:** state $|0\rangle_1 \ldots |0\rangle_n$ has
$(1,0)$ exponential decay of correlations

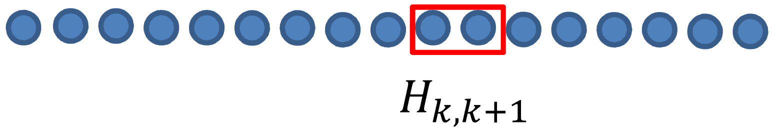
**Example 2:** state $|\phi_+\rangle_{12}|\phi_+\rangle_{34} \ldots |\phi_+\rangle_{n-1,n}$ has
$(2,2)$ exponential decay of correlations




For arbitrary lattice, $l$ is distance between sets.

# Which states obey exp. decay of correlations?

Consider systems with local Hamiltonian: $H = \sum_k H_{k,k+1}$




Ground state: $|\psi_0\rangle : \ H|\psi_0\rangle = E_0|\psi_0\rangle$

Spectral gap: $\Delta(H) = E_1 - E_0$

Condensed matter physics folklore:

| **Model** | **Spectral gap** | **Ground state** |
|---|---|---|
| non-critical $\Leftrightarrow$ | gapped | $\Rightarrow$ Exp. decay of correlations |
| critical $\Leftrightarrow$ | non-gapped | $\Leftarrow$ Long range correlations |

Rigorous resutls

(Araki, Hepp, Ruelle '62, Fredenhagen '85) Ground states in relativistic systems

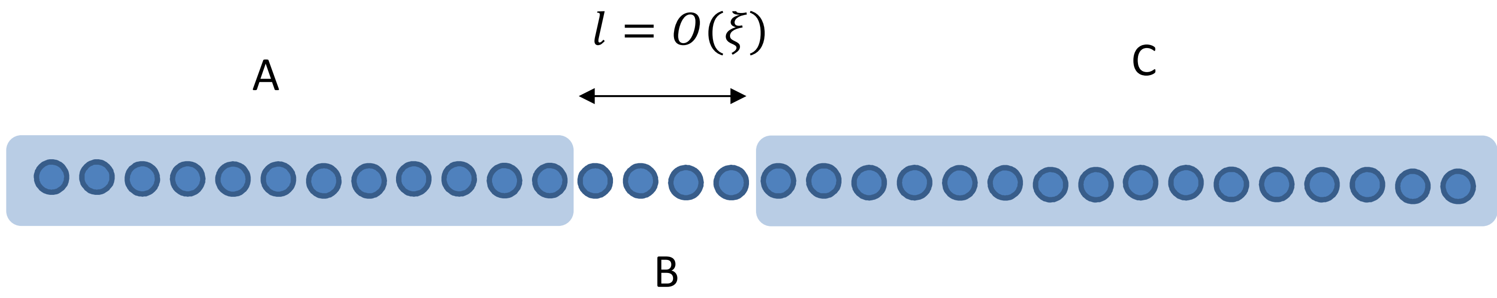
(Araki '69) Thermal states of 1D local Hamiltonians

(Hastings '04, Nachtergaele, Sims '06, Koma '06) Ground states of gapped local Hamiltonians
(Analytic proof: Lieb-Robinson bounds, etc…)

(Araronov, Arad, Landau, Vazirani '10) Ground states of gapped frustration-free local Hamiltonians
(Combinatorial Proof: Detectability Lemma)

# Exp. decay of correlations vs. Area Law

Exponential decay of correlations suggests Area Law:



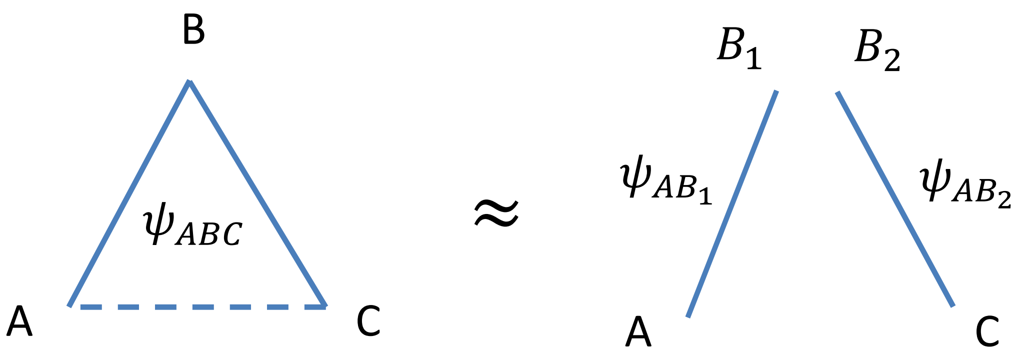

EDC implies ~~$||\rho_{AC} - \rho_A \otimes \rho_C|| \leq 2^{-l/\xi}$~~

Then by Uhlmann theorem we can represent $B$ as $B_1 \otimes B_2$, in such a way that

$S(A) = S(B_1) \leq O(\xi)$

$$\psi_{ABC} \approx \psi_{AB_1} \otimes \psi_{CB_2}$$

Area Law!




Counterexample:
hiding states

# Data hiding states

D.P. DiVincenzo, D.W. Leung, and B.M. Terhal, IEEE Trans. Inf. Theo. **48**, 580 (2002); quant-ph/0103098]

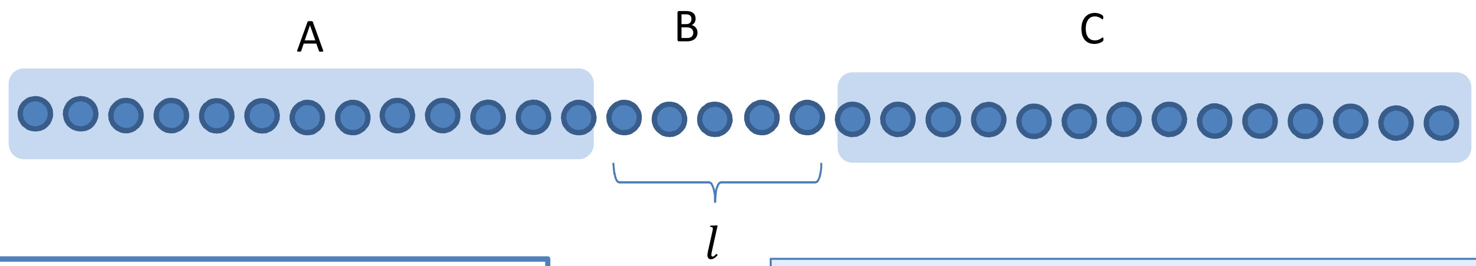
There are states which:

- Cannot be distinguished from maximally mixed states by LOCC
- Are far from maximally mixed states in trace norm

Are these some pathological states of very particular weird construction?

## A generic state is of this kind!

[P. Hayden, D.W. Leung, and A. Winter. Comm. Math. Phys. **265**, 95 (2006), quant-ph/0407049]




$\psi_{ABC}$ drawn from Haar measure
if $\mathrm{size}(A) \approx \mathrm{size}(C)$ then w.h.p.

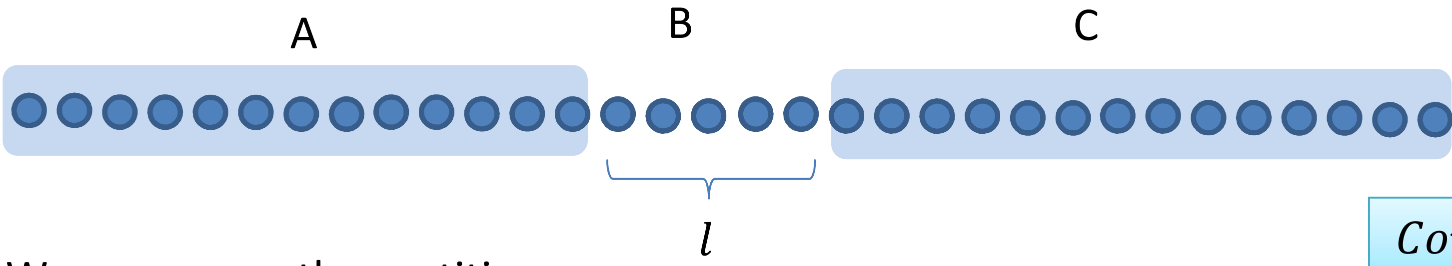
- $Cor(A{:}C) = 2^{-\Omega(l)}$
- $S(A) \approx S(C) \approx \frac{n}{2} - l$

**Conlusion**: small correlations in fixed partition do not imply Area Law.

**But**: we can move te partitions!
Because we assume EDC in every partition

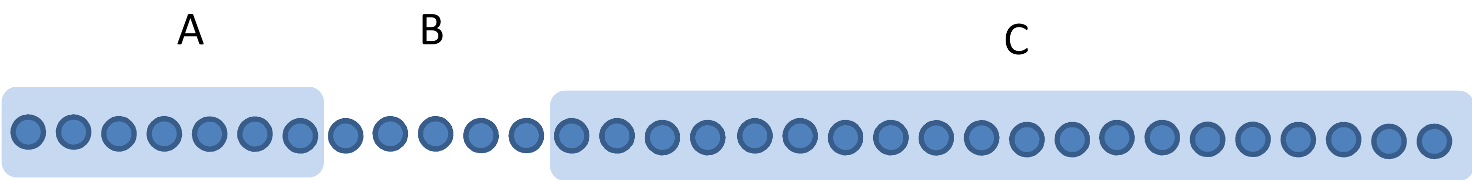
# Random states are not a counterexample

Random states **do not** satisfy EDC: they have **large correlations** in some partitions




$Cor(A{:}C) \sim 2^{-l}$

We now move the partition:

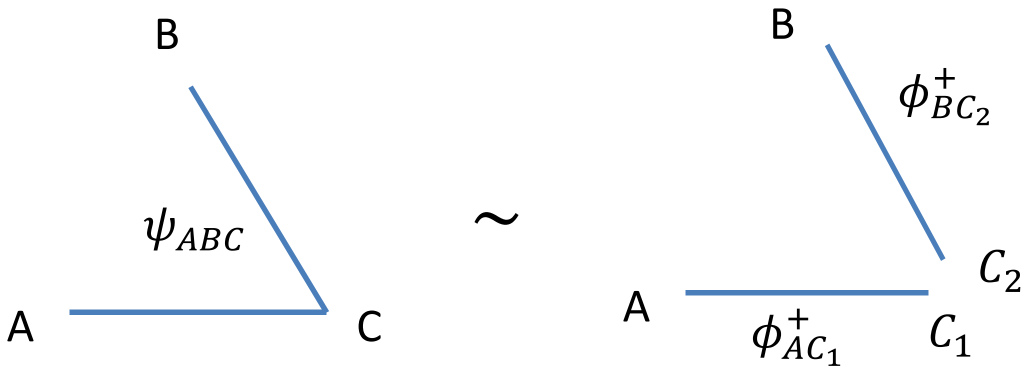



if $\text{size}(AB) < \text{size}(C)$ then w.h.p. $\rho_{AB} \approx \tau_{AB}$ where $\tau_{AB} = \frac{I_{AB}}{|A||B|} = \tau_A \otimes \tau_B$

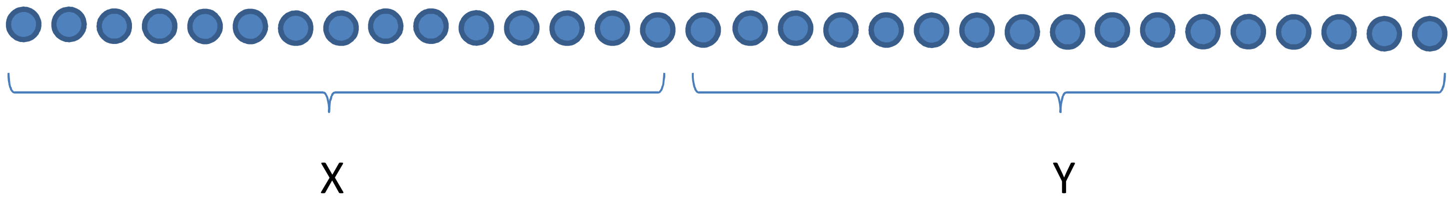
A and B are completely decoupled!

$$\psi_{ABC} \approx \phi^{+}_{AC_1} \otimes \phi^{+}_{AC_2}$$

Large correlations: $Cor(A{:}C) \approx 1$



# Exp. decay of correlations implies Area Law




**Theorem (Brandao, H. 2012):** If $|\psi\rangle_{1\ldots n}$ has exponential decay od correlations which

- has the range $\xi$
- obeys for distances $l \geq l_0$

then for every subsystem $X = [1, \ldots, r]$ we have

$$S(X) \leq l_0 2^{O(\xi \, log \xi)} = const$$

- Actually we proved more:

$$H_{max}^{2^{-l}}(X) \leq l_0 2^{O(\xi log \xi)} + l$$

- We proved it only in 1D

For the purpose of the talk, I will take $l_0 = 1$

Solely information-theoretic result - no dynamics, no Hamiltonian!

# A vulgar summary

- Condensed Matter (CM) community always knew that EDC implies Area Law.
- Quantum Information (QI) community provided a counterexample (hiding states)
- QI people removed the trouble they caused themselves (this talk)
- CM community apparently didn't noticed either of this minor perturbations

| EDC $\Rightarrow$ Area Law stays true! |
| --- |

Moreover:

- First completely information theoretic result in this area
  - No dynamics, no Hamiltonian
- Proves rigorously that states with EDC are not complex
  - ⇓ Quantum computation must use **critical phase**

# Correlations: definition and properties

$$Cor(X:Y) = \max_{||M|| \leq 1, ||N|| \leq 1} |tr\big(M \otimes N(\rho_{XY} - \rho_X \otimes \rho_Y)\big)|$$

here $\|M\|$ – is operator norm of M

It is closely related to standard two point correlation function:

$$Cor(X:Y) = \max_{\mathrm{M,N}}(\langle M \otimes N\rangle - \langle M\rangle\langle N\rangle)$$

**Note**:
This quantification correlation does not involve communication before measurement.

A version with communication:

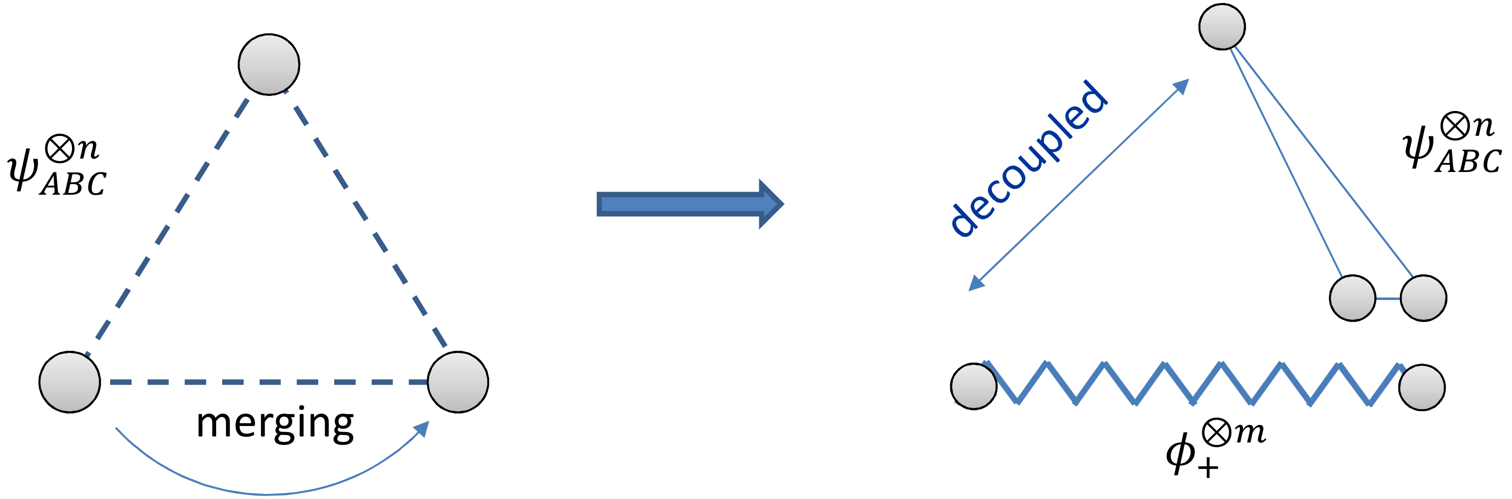
$$Cor^{\rightarrow}(X:Y) = \max_{0 \leq A_k \leq I, \|B_k\| \leq 1} \sum_{k=1}^{n} |\mathrm{tr}((A_k \otimes B_k)(\rho_{XY} - \rho_X \otimes \rho_Y))|$$

**Useful bound:**

$$\mathrm{Cor} \geq \frac{Cor^{\rightarrow}}{\text{\# outcomes to communicate}}$$

# State merging

H., Oppenheim, Winter, 2005



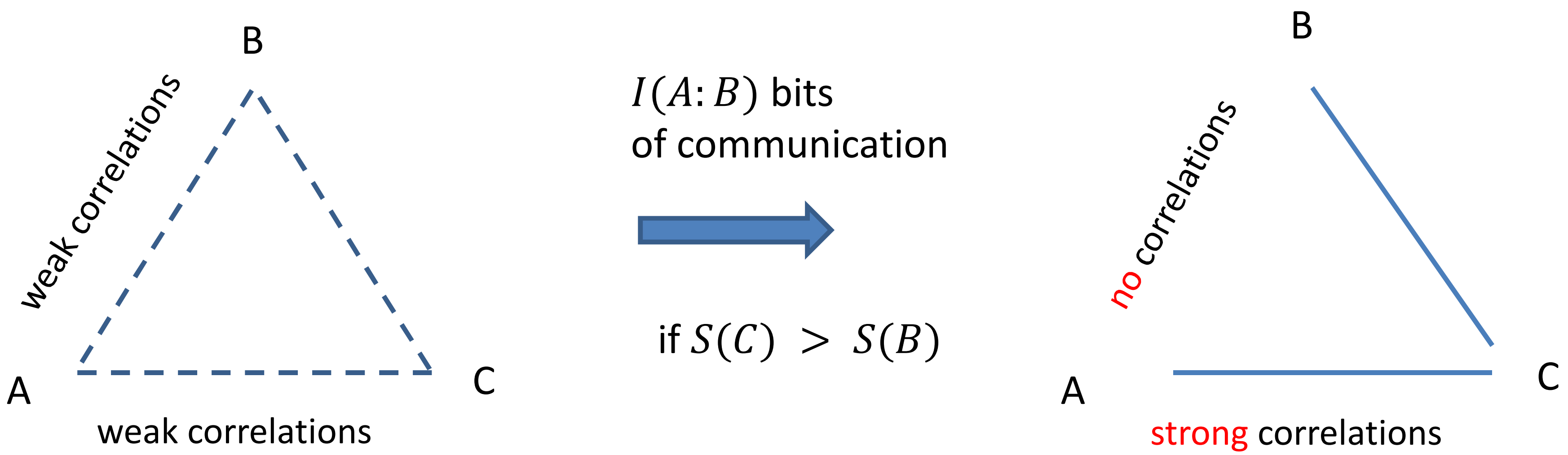

$I(A{:}B)$ bits of communication / copy

$m = S(C) - S(B)$ pairs of **maximally entangled** states / copy


weak correlations

strong correlations

# A primitive



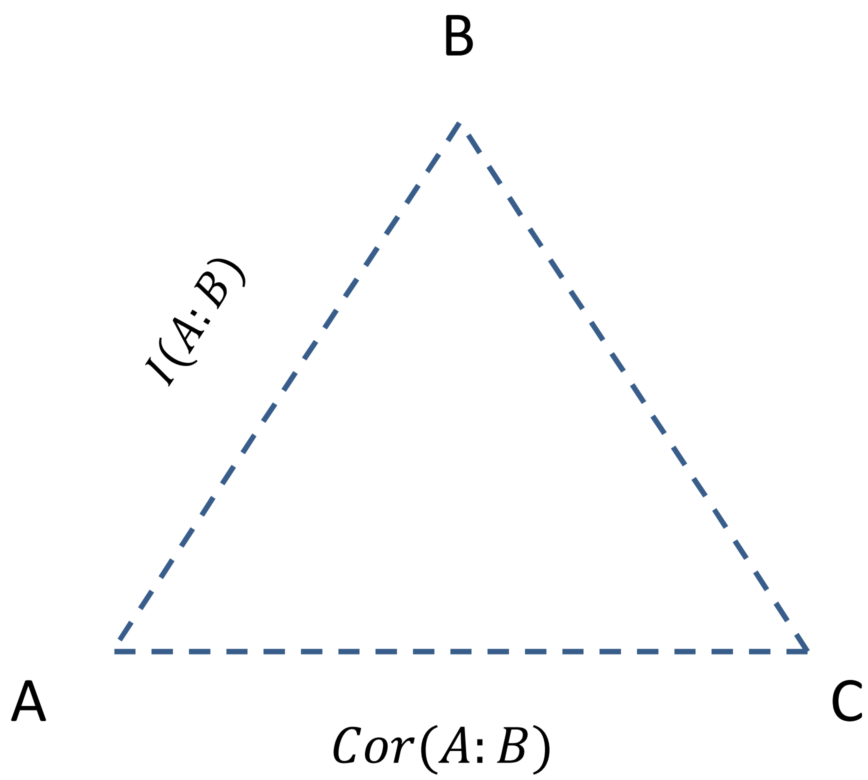

Suppose, that $S(C) \geq S(B) + 1$

Then we obtain maximally entangled pair: $Cor^{\rightarrow}(A: C) = 1$

After paying the price of communication: $Cor(A: C) \geq 2^{-I(A:B)}$

**Factoid®**: If correlations between A and B are less than $2^{-I(A:B)}$ then $S(C) \leq S(B) + 1$

# A primitive




If

- $I(A:B)$ is small
- $Cor(A:C)$ is very small

then

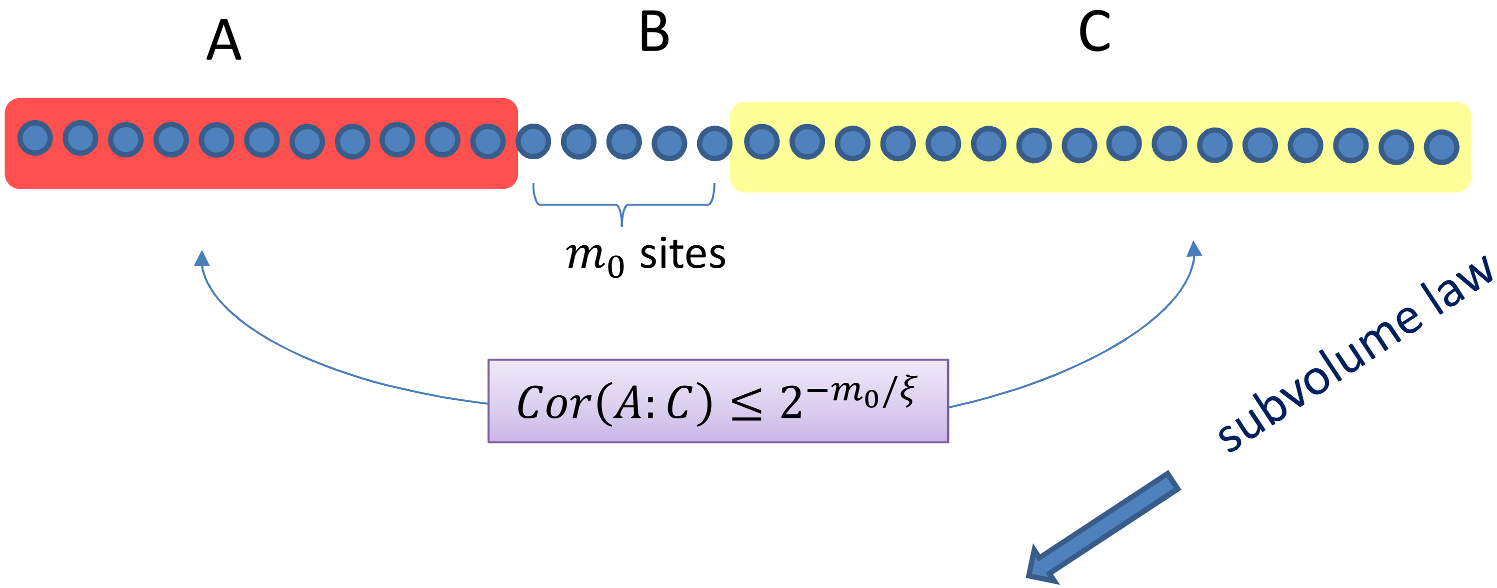
entropy of C cannot be much larger than entropy of B

$$Cor(A:B) < 2^{-I(A:B)} \quad \Rightarrow \quad S(C) < S(B) + 1$$

# From subvolume law to area law

Since $I(A:B) \leq 2S(B)$ we obtain

**Factoid®**: If correlations between A and B are less than $2^{-2S(B)}$ then $S(C) \leq S(B) + 1$




Suppose that the region B satisfies $S(B) < \epsilon\, m_0$ with $\epsilon = \frac{1}{2\xi}$

We get: $Cor(A:C) \leq 2^{-\frac{m_0}{\xi}} < 2^{-2S(B)}$

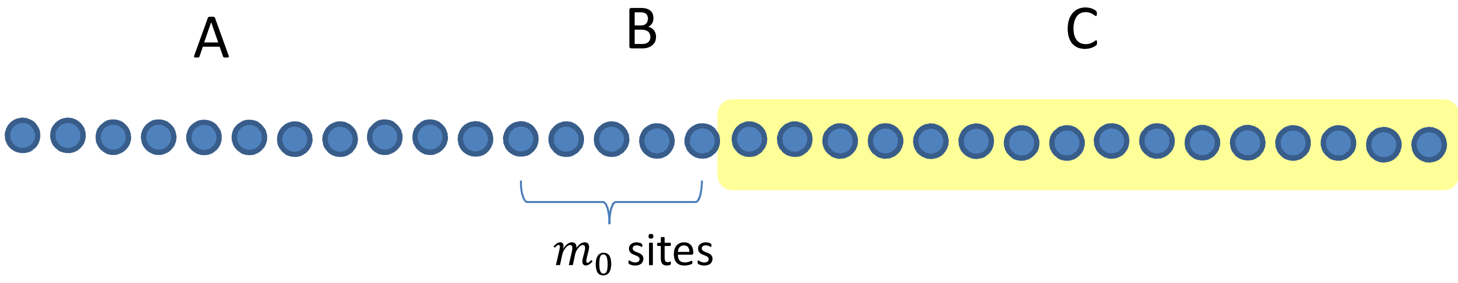
Area Law!

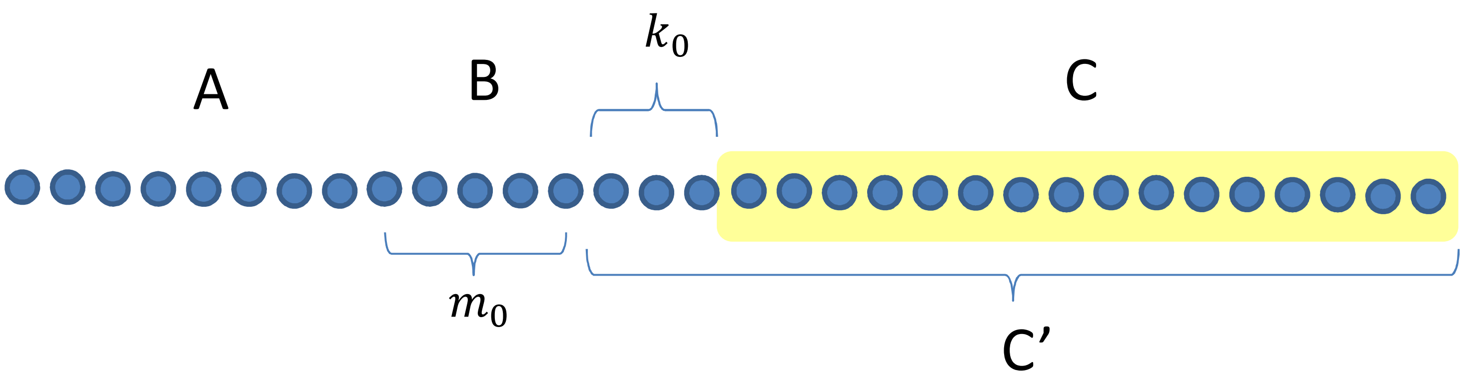
And by Factoid: $S(C) < \frac{1}{2\xi} m_0 + 1 = \text{const}$

# From subvolume law to Area Law: conclusion

From previous slides we conclude:

**Conclusion:** If nearby **C** we will find region B with $S(B) \leq \frac{1}{2\xi}|B|$ then we obtain Area Law for B.




$$S(C) \leq S(C') + k_0 \leq S(B) + 1 + k_0 = const$$

# Subvolume for mutual information

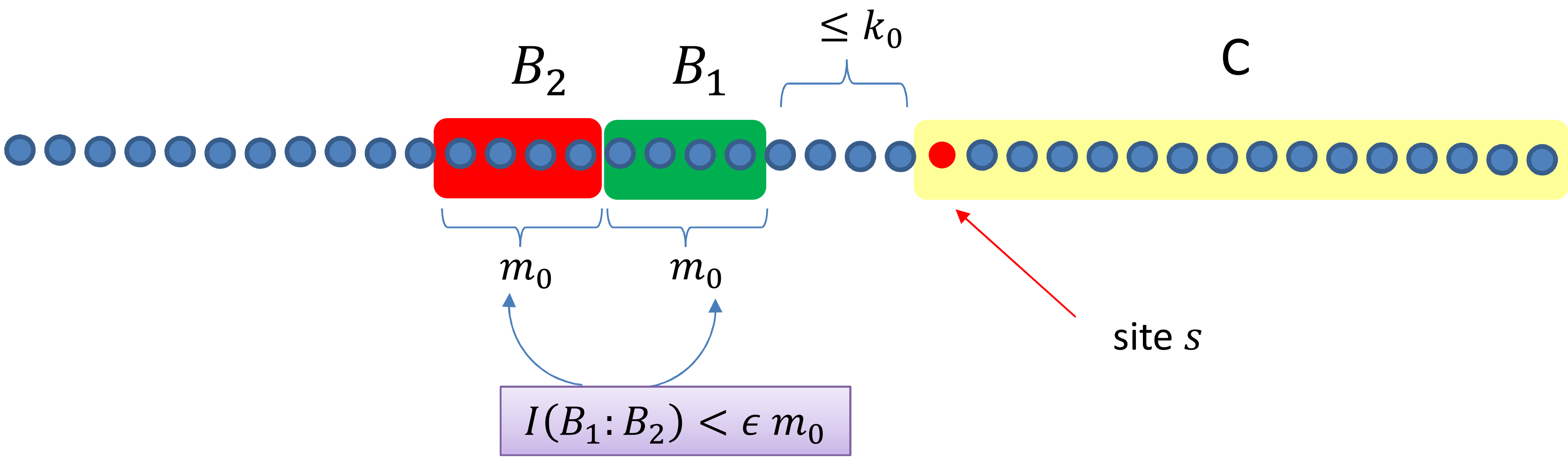
**Saturation lemma [Hastings]**
Suppose we start from arbitrary site $s$. Then for arbitrary $\epsilon$ we can find regions $B_1$ and $B_2$

- each of length $m_0 = 2^{O(1/\epsilon)}$
- at a distance from the site $s$

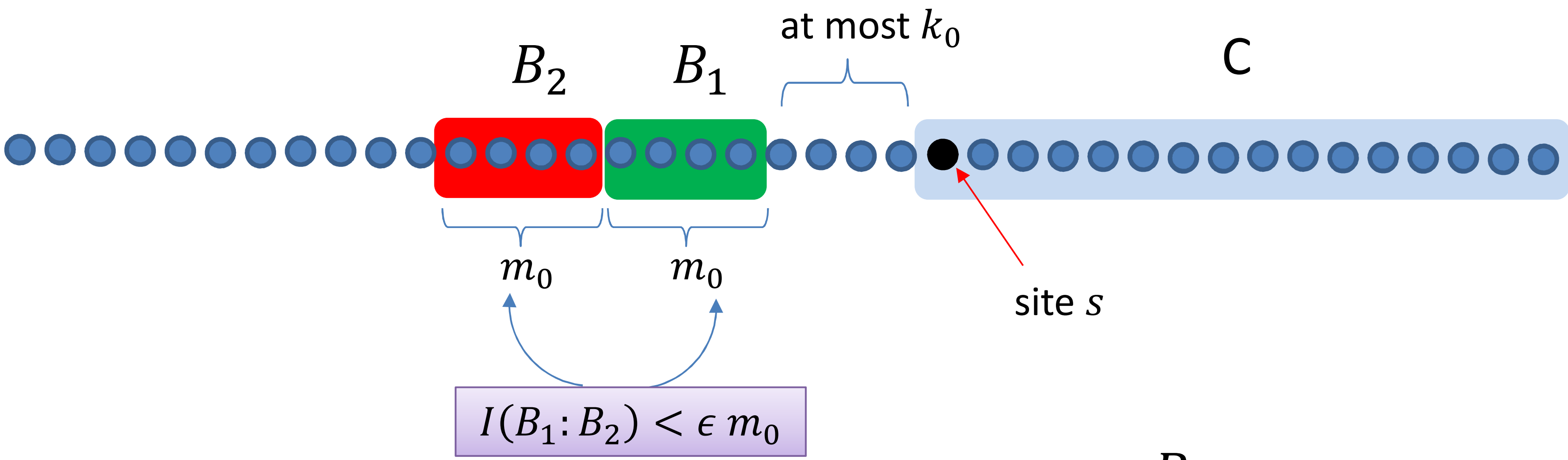
$$k_0 = 2^{O(1/\epsilon)}$$

- such that $I(B_1 : B_2) < \epsilon\, m_0$

**Remark**: if needed, we can make $m_0$ larger at the expence of $k_0$.



# From subvolume of mutual information to subvolume of entropy




We want $S(B_1) < \epsilon' m_0$.

But:

$$S(B_1) = \underbrace{I(B_1{:}B_2)}_{< \epsilon\, m_0} + \underbrace{[S(B_1B_2) - S(B_2)]}_{\leq 1}$$

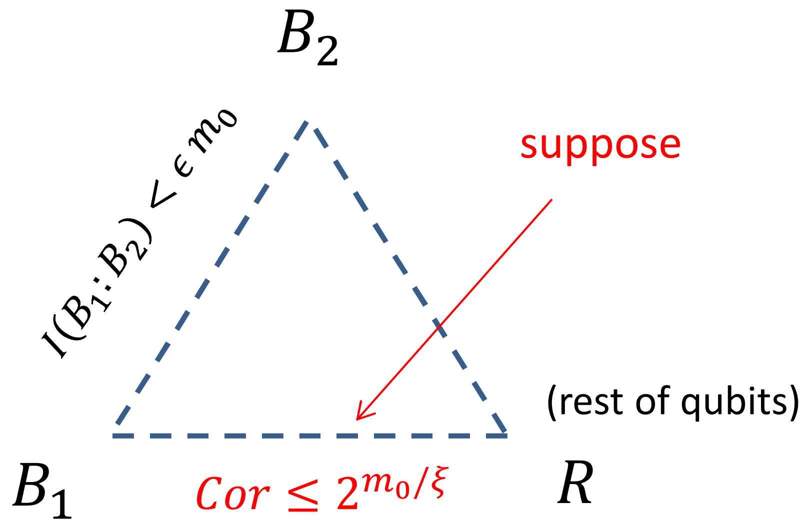
suppose

?

Then we would be done:

$$S(B_1) < \epsilon\, m_0 + 1 \leq 2\epsilon\, m_0$$

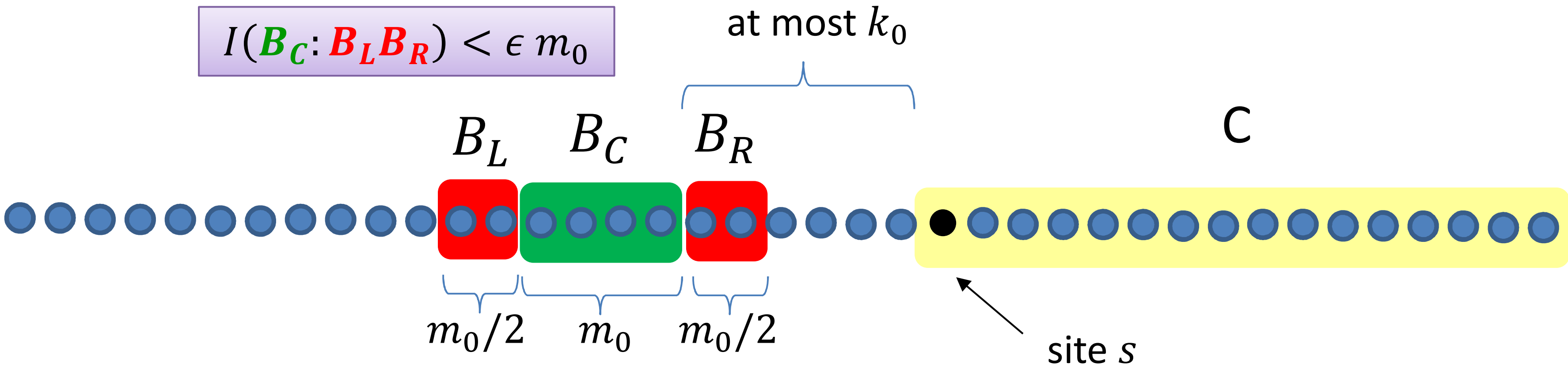
if we take $m_0$ suitably large.

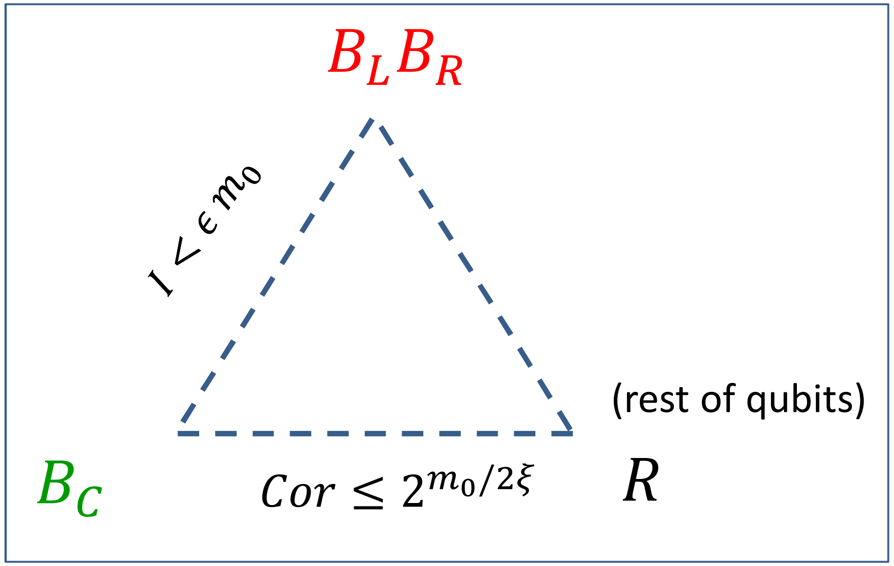



Then, merging not possible. hence

$$S(R) - S(B_2) < 1$$

# From subvolume of mutual information to subvolume of entropy




If we take $\epsilon < \frac{1}{2\xi}$ then

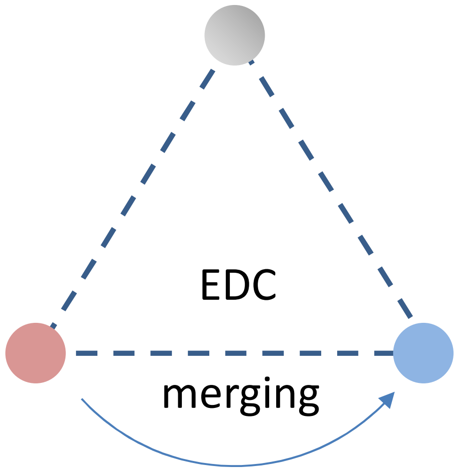
$$Cor(B_C : R) < 2^{-I(B_C : B_L B_R)}$$

hence by Factoid we obtain

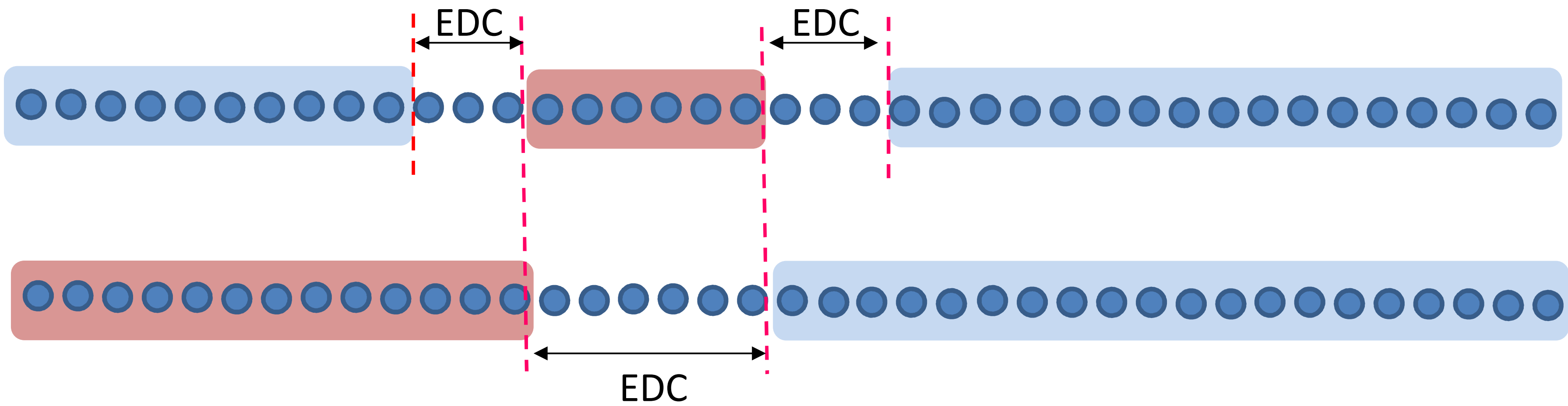
$$S(R) - S(B_C) < 1$$

$$S(B_C) = \overbrace{I(B_C : B_L B_R)}^{<\,\epsilon\, m_0} + \overbrace{[S(R) - S(B_C)]}^{<\,1} \quad \Rightarrow \quad S(B_C) < 2\epsilon\, m_0$$

# Where we used merging and where we used EDC




Saturation of mutual information → subvolume law for entropy




Subvolume law for entropy → Area Law