Max- Relative Entropy: a parent quantity in one-shot information theory

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Quantum Relative Entropy

- a fundamental quantity in Quantum Mechanics & Quantum Information Theory :

  The quantum relative entropy of a state ρ w.r.t. a positive operator σ :

$$D(\rho \| \sigma) := \mathrm{Tr}\left(\rho \log \rho\right) - \mathrm{Tr}\left(\rho \log \sigma\right)$$

$\log \equiv \log_2$

  well-defined if $\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$

- It acts as a parent quantity for the *von Neumann entropy*:

$$S(\rho) := -\mathrm{Tr}\left(\rho \log \rho\right) = -D(\rho \| I)$$

$(\sigma = I)$

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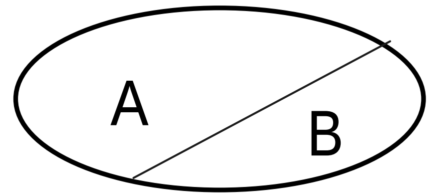
- It also acts as a parent quantity for other entropies:

e.g. for a bipartite state ρ_{AB} :



- *Conditional entropy*

$$S(A|B) := S(\rho_{AB}) - S(\rho_B) \quad = -D(\rho_{AB} \| I_A \otimes \rho_B)$$

- *Mutual information*

$$\rho_B = \mathrm{Tr}_A\ \rho_{AB}$$

$$I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \ = D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

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Quantum relative entropy as a "distance" measure between states

"distance"

~~*symmetric*~~
~~*triangle inequality*~~

$$D(\rho \| \sigma) \geq 0 \quad \text{if} \quad \rho, \sigma \ \text{states}$$

$$= 0 \text{ if \& only if } \rho = \sigma$$

- Monotonicity of Quantum Relative Entropy under a linear completely positive trace-preserving (CPTP) map Λ :

$$D(\Lambda(\rho) \| \Lambda(\sigma)) \leq D(\rho \| \sigma) \qquad \text{..........(1)}$$

- Invariance under joint unitaries:

$$D(U\rho U^{\dagger} \| U\sigma U^{\dagger}) = D(\rho \| \sigma)$$

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Outline

- *Define a generalized relative entropy: Max-relative entropy*

  --- a parent quantity for the min-entropies of Renner

  - *Discuss its properties and operational interpretations*

- *Define its smoothed version: smooth max-relative entropy*

  - *Discuss its significances in one-shot information theory*

- *Max-relative entropy* → *entanglement monotone*

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- *Definition :* The max- relative entropy of a state ρ & a positive operator σ is

$$D_{\max}(\rho \| \sigma) := \inf \left\{ \gamma : \rho \leq 2^{\gamma} \sigma \right\}$$

$\text{supp}\, \rho \subseteq \text{supp}\, \sigma$

$$(2^{\gamma} \sigma - \rho) \geq 0$$

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- *Definition :* The max- relative entropy of a state ρ & a positive operator σ is

$$D_{\max}(\rho \| \sigma) := \inf \left\{ \gamma : \rho \leq 2^{\gamma} \sigma \right\}$$

- Equivalently, $\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$

$$D_{\max}(\rho \| \sigma) := \inf \left\{ \gamma : \operatorname{Tr} P^{\gamma} (\rho - 2^{\gamma} \sigma) = 0 \right\}$$

P^{γ} : *the projector onto eigenspace of* $(\rho - 2^{\gamma} \sigma)$ *corrs. to non-negative eigenvalues*

- *Definition :* The max- relative entropy of a state ρ & a positive operator σ is

$$D_{\max}(\rho\|\sigma) := \inf\{\log\lambda : \rho \leq \lambda\sigma\}$$

- Equivalently, $\mathrm{supp}\,\rho \subseteq \mathrm{supp}\,\sigma$

$$D_{\max}(\rho\|\sigma) := \inf\{\gamma : \mathrm{Tr}\, P^{\gamma}(\rho - 2^{\gamma}\sigma) = 0\}$$

$$D_{\max}(\rho\|\sigma) = -\log\left(\lambda_{\max}(\sigma^{-1/2}\rho\sigma^{-1/2})\right)$$

pseudoinverse

- *asymmetric measure*

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Symmetrising the max-relative entropy

(well-known in the mathematical literature)

- Definition 1: The Thompson metric for 2 positive operators ρ and σ is given by:

$$d_T(\rho,\sigma) = \max\left\{D_{\max}(\rho\,\|\,\sigma), D_{\max}(\sigma\,\|\,\rho)\right\}$$

- Definition 2: Hilbert's projective metric for 2 positive operators ρ and σ is given by:

$$d_H(\rho,\sigma) = D_{\max}(\rho\,\|\,\sigma) + D_{\max}(\sigma\,\|\,\rho)$$

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Properties of max-relative entropy

- *Non-negativity:*

$$D_{\max}(\rho\|\sigma) \geq 0 \quad \text{if} \quad \rho, \sigma \quad \text{are states}$$

- *Relation with quantum relative entropy:*

$$D_{\max}(\rho\|\sigma) \geq D(\rho\|\sigma)$$

- *Monotonicity under CPTP maps:*

$$D_{\max}(\Lambda(\rho)\|\Lambda(\sigma)) \leq D_{\max}(\rho\|\sigma) \quad \forall \quad \text{CPTP map } \Lambda$$

- *Invariance under joint unitaries:*

$$D_{\max}(U\rho U^{\dagger}\|U\sigma U^{\dagger}) = D_{\max}(\rho\|\sigma)$$

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Properties of max-relative entropy contd.

- *Quasi-convexity:*

  For two mixtures of states $\rho = \sum_{i=1}^{n} p_i \rho_i$ & $\sigma = \sum_{i=1}^{n} p_i \sigma_i$

$$D_{\max}(\rho \| \sigma) \leq \max_{1 \leq i \leq n} D_{\max}(\rho_i \| \sigma_i)$$

In contrast, the quantum relative entropy is jointly convex in its arguments

$$D(\rho \| \sigma) \leq \sum_{i=1}^{n} p_i D(\rho_i \| \sigma_i)$$

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$D_{\max}(\rho \| \sigma)$: a *parent* for Renato's min- entropies

Just as:

von Neumann entropy

$$S(\rho) = -D(\rho \| I)$$

$$(\sigma = I)$$

Min-entropy

[Renner]

$$H_{\min}(\rho) := -D_{\max}(\rho \| I)$$
$$= -\log \lambda_{\max}(\rho)$$

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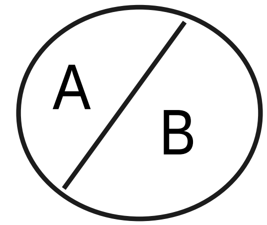
For a bipartite state ρ_{AB} :



- *Conditional min-entropy* *[Renner]*

$$H_{\min}(A|B)_\rho := \max_{\sigma_B}\{-D_{\max}(\rho_{AB} \| I_A \otimes \sigma_B)\}$$

just as: *Quantum conditional entropy*

$$S(A|B) = -D(\rho_{AB} \| I_A \otimes \rho_B) = \max_{\sigma_B}\{-D(\rho_{AB} \| I_A \otimes \sigma_B)\}$$

- *Max-information* *[Berta, Christandl, Renner]*

$$I_{\max}(A:B)_\rho := \min_{\sigma_B} D_{\max}(\rho_{AB} \| \rho_A \otimes \sigma_B)$$

just as: *Quantum mutual information* *[Buscemi & ND]*

$$I(A:B) = D(\rho_{AB} \| \rho_A \otimes \rho_B) = \min_{\sigma_B} D(\rho_{AB} \| \rho_A \otimes \sigma_B)$$

Operational interpretations of the max-relative entropy (i)

- *Theorem 1 [T.Rudolph, R.Spekkens]:*

  The maximum probability with which a state ρ can appear in a convex decomposition of a state σ is:

  $$= 2^{-D_{\max}(\rho\|\sigma)} \quad \textit{if} \quad \operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$$

  $$= 0 \quad \textit{else}$$

- since $D_{\max}(\rho\|\sigma) = -\log\left(\lambda_{\max}(\sigma^{-1/2}\rho\sigma^{-1/2})\right)$

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Operational interpretations of the max-relative entropy (ii)

- *Separability of a bipartite state*

[Lewenstein & Sanpera] :The state σ of any bipartite system can always be written as a weighted average of a separable state ρ and another (possibly entangled) state ω ,

$$\sigma = \lambda\rho + (1-\lambda)\omega$$

such that the weight λ of ρ is maximal.

ρ : Best separable approximation (BSA) of the state σ

λ : separability of the state σ *[Wellen & Kus]*

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- *Theorem 2 [ND,T.Rudolph]:*

The separability of the state σ of a bipartite system is given by:

$$\lambda = \max_{\rho \in \mathcal{S}(\mathcal{H})} 2^{-D_{\max}(\rho \| \sigma)}$$

set of separable states

Proof: Follows from Theorem 1.

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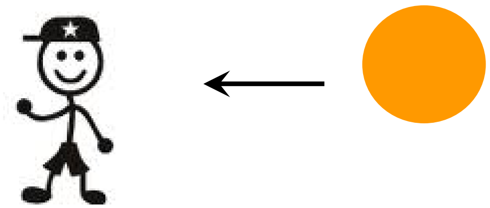
Operational interpretations of the max-relative entropy (iii)

- *Multiple state discrimination problem:*



Bob

a quantum system

& told

its state	with prob.
ρ_1	$1/M$
ρ_2	⋮
⋮	
ρ_M	$1/M$

- He does measurements to infer the state: **POVM**

$$\{E_1,..,E_M\}: \ 0 \le E_i \le I; \ \sum_{i=1}^{M} E_i = I$$

- *His optimal average success probability:*

$$p^*_{succ} := \max_{\{E_1,..,E_k\}} \frac{1}{M} \sum_{i=1}^{M} \mathrm{Tr}\left(E_i \rho_i\right)$$

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- *Theorem 3 [M.Mosonyi & ND]:*

  The optimal average success probability in this multiple state discrimination problem is given by:

$$p^*_{succ} = \frac{1}{M} \min_{\sigma} \max_{1 \le i \le M} 2^{D_{\max}(\rho_i \| \sigma)}$$

Proof: follows from a lemma by [Koenig, Renner, Schaffner].

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Operational interpretations of the max-relative entropy (iv)

- *Binary hypothesis testing problem:*

  - A quantum system is either in a state ρ or in a state σ
  - Alice is given multiple, identical copies of the system
    - The state of the $n-$copy system is $\rho_n = \rho^{\otimes n}$ or $\sigma_n = \sigma^{\otimes n}$
  - A **test** on the $n-$copy system: $T \in \mathcal{B}(\mathcal{H}^{\otimes n})$

  $0 \leq T \leq I_n$, which determines the binary POVM

  $$\{T, I_n - T\}$$

In the language of hypothesis testing

- *null hypothesis* $H_0 : \rho_n = \rho^{\otimes n}$
- *alternative hypothesis* $H_1 : \sigma_n = \sigma^{\otimes n}$

- If the outcome corrs. to T occurs -- we accept H_0

  else -- we accept H_1

-

Possible errors	*inference*	*actual state*
Type I	σ_n	ρ_n
Type II	ρ_n	σ_n

Prob(Type I error) $= \alpha_n(T) = \mathrm{Tr}((I_n - T)\rho_n)$

Prob(Type II error) $= \beta_n(T) = \mathrm{Tr}(T\sigma_n)$

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$$\alpha_n(T) + \beta_n(T) > 0 \quad \forall\, T,$$ unless $\text{supp}\,\rho \perp \text{supp}\,\sigma$

- Define: $e_n := \min_T \{\alpha_n(T) + \beta_n(T)\}$

Note: $2e_n =$ *min. avg. error in symmetric hypothesis testing*

$$-\lim_{n\to\infty} \frac{1}{n} \log e_n = C(\rho \| \sigma)$$

[Audenaert et al]

where $C(\rho \| \sigma) := -\inf_{0\le t\le 1} \log\left(\text{Tr}(\rho^t \sigma^{1-t})\right)$

Chernoff distance

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- It is known that *[Audenaert, Mosonyi, Verstraete]*

$\forall\ n \in \mathrm{N}$ $$-\frac{1}{n}\log e_n \leq C(\rho \| \sigma)$$

where

$$e_n = \min_T \{\alpha_n(T) + \beta_n(T)\}$$

$C(\rho \| \sigma) =$ *Chernoff distance*

- *Theorem 5 [ND & T.Rudolph]:*

$\forall\ n \in \mathrm{N}$ $$-\frac{1}{n}\log e_n \geq D_{\max}(\rho \| \sigma)$$

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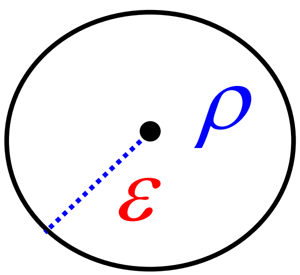
Smooth max-relative entropy

$$D_{\max}^{\varepsilon}(\rho\|\sigma) := \min_{\bar{\rho}\in B^{\varepsilon}(\rho)} D_{\max}(\bar{\rho}\|\sigma)$$

$$B^{\varepsilon}(\rho) := \left\{\bar{\rho}\geq 0, \mathrm{Tr}\bar{\rho}=1:\ \sqrt{1-F(\rho,\bar{\rho})}\leq\varepsilon\right\}$$

fidelity $= \mathrm{Tr}\sqrt{\sqrt{\rho}\,\bar{\rho}\,\sqrt{\rho}}$



Smooth max-relative entropy contd.

Relation with other generalized relative entropies

(I) Relation with the sup-spectral divergence rate

Information spectrum approach

- Developed in Classical Info. theory by Verdu and Han
- first extended to Quantum Info. theory by Hayashi, Nagaoka & Ogawa.
- a powerful method for obtaining the optimal rates of various protocols.
- The power of the method : it does not rely on any specific nature of the sources, channels etc.

  (no i.i.d. assumption)

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Information spectrum method

- Two fundamental quantities (generalized relative entropies)

$\overline{D}(\hat{\rho}\,\|\,\hat{\sigma})$: sup-spectral divergence rate

$\underline{D}(\hat{\rho}\,\|\,\hat{\sigma})$: inf-spectral divergence rate

$\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ arbitrary sequence of states;

$\hat{\sigma} = \{\sigma_n\}_{n=1}^{\infty}$ arbitrary sequence of pos. ops.;

$$\overline{D}(\hat{\rho}\,\|\,\hat{\sigma}) := \inf\left\{\gamma : \lim_{n\to\infty} \mathrm{Tr}\left[P_n^{\gamma}(\rho_n - 2^{n\gamma}\sigma_n)\right] = 0\right\}$$

P_n^{γ} : the projector onto the eigenspace of non-neg. eigenvalues of $(\rho_n - 2^{n\gamma}\sigma_n)$

$$D_{\max}(\rho\,\|\,\sigma) := \inf\left\{\gamma : \mathrm{Tr}\, P^{\gamma}(\rho - 2^{\gamma}\sigma) = 0\right\}$$

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(I) Relation with the sup-spectral divergence rate

- *Theorem 6 [ND]:*

$$\bar{D}(\hat{\rho} \| \hat{\sigma}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho_n \| \sigma_n)$$

where $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ and $\hat{\sigma} = \{\sigma_n\}_{n=1}^{\infty}$

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(I) Relation with the quantum relative entropy

- *Lemma 1:*

If $\hat{\rho} = \left\{\rho^{\otimes n}\right\}_{n=1}^{\infty}$ and $\hat{\sigma} = \left\{\sigma^{\otimes n}\right\}_{n=1}^{\infty}$

$$\bar{D}\left(\hat{\rho} \,\|\, \hat{\sigma}\right) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho_n \,\|\, \sigma_n) \equiv D(\rho \,\|\, \sigma)$$

Quantum Stein's lemma

[Hiai & Petz; Hayashi & Nagaoka]

(II) Relation with quantum relative entropy

- *Lemma 2 [ND, Mosonyi, Hsieh, Brandao]:*

- If $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB});\ \sigma_B \in \mathcal{D}(\mathcal{H}_B);\ \text{supp}\ \rho_A \subseteq \text{supp}\ \sigma_A;$

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \min_{\omega_n \in \mathcal{D}(\mathcal{H}_B)} \frac{1}{n} D_{\max}^{\varepsilon}(\rho_{AB}^{\otimes n} \| \sigma_A^{\otimes n} \otimes \omega_n)$$

$$= D(\rho_{AB} \| \sigma_A \otimes \rho_B)$$

Proof: Follows from the Generalized Quantum Stein's lemma [Brandao & Plenio]

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(II) Relation with relative Renyi entropy

- *Lemma 3 [Mosonyi, ND]:*

If $[\rho,\sigma]=0,$ then $D_{\max}(\rho\|\sigma)=D_{\infty}(\rho\|\sigma)$

If $[\rho,\sigma]\neq 0,$ then $D_{\max}(\rho\|\sigma)\neq D_{\infty}(\rho\|\sigma)$ in general

$$D_{\infty}(\rho\|\sigma):=\lim_{\alpha\to\infty} D_{\alpha}(\rho\|\sigma);$$

relative Renyi entropy
of order $\alpha\neq 1$

$$D_{\alpha}(\rho\|\sigma):=\frac{1}{\alpha-1}\log \operatorname{Tr} \rho^{\alpha}\sigma^{1-\alpha}$$

$$D_{\max}(\rho\|\sigma)\geq D_{2}(\rho\|\sigma)$$

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Operational significances of the

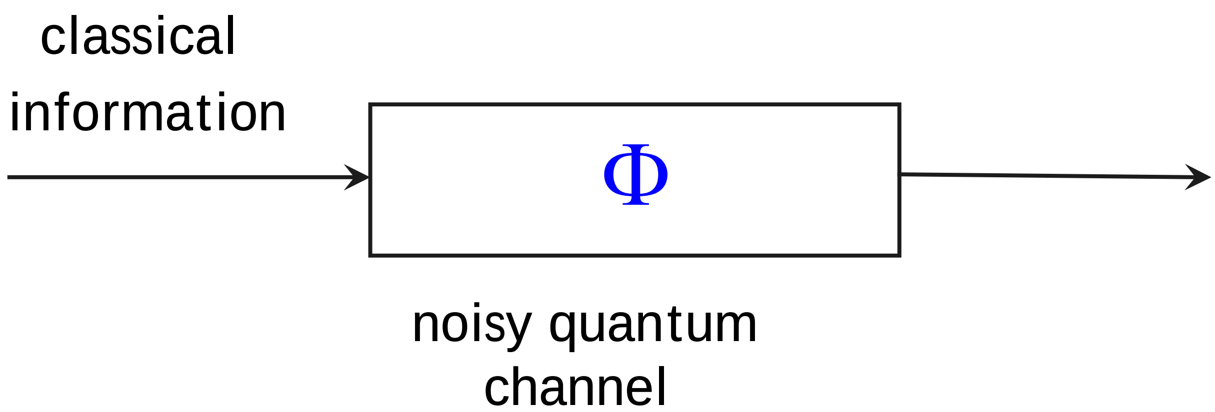
smooth max-relative entropy

in one-shot information theory

(a) *Transmission of classical information through a quantum channel*

(b) *Binary hypothesis testing*

(a) Transmission of Classical Info through a quantum channel



- Asymptotic i.i.d scenario
- $n-$independent uses of the channel $:\Phi^{\otimes n}$

memoryless

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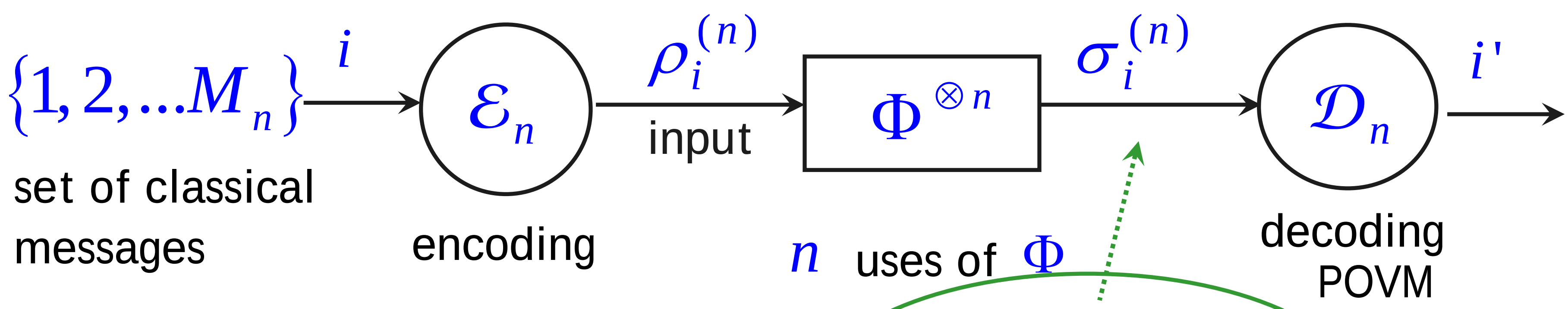
■ Transmission of Classical Info through a *memoryless* quantum channel



$$\sigma_i^{(n)} := \Phi^{\otimes n}\left(\rho_i^{(n)}\right)$$

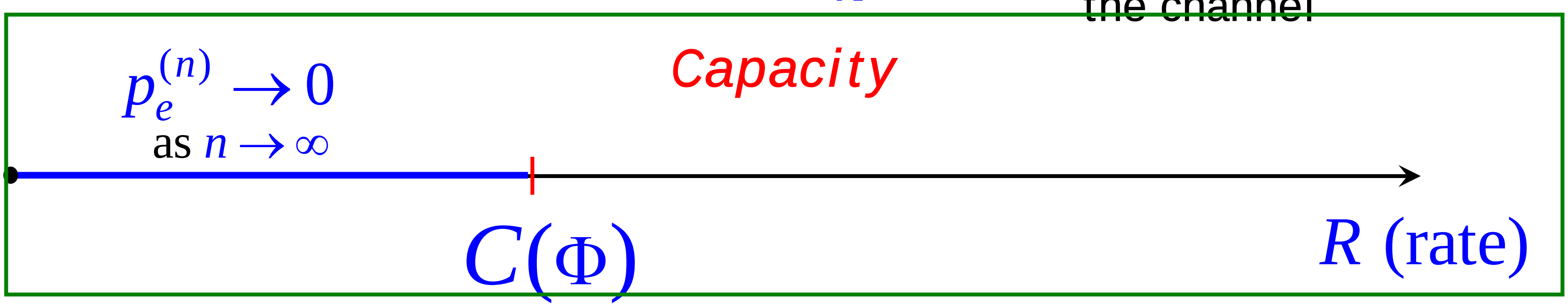
channel output

$$C_n := (\mathcal{E}_n, \mathcal{D}_n, M_n)$$

(code)

■ Rate of the code : $R = \dfrac{\log M_n}{n}$

no. of bits of message transmitted per use of the channel



(I) Product-state classical capacity $C_p(\Phi)$

- Encoding restricted to product states, i.e.,

$$\mathcal{E}_n : \qquad i \to \rho_i^{(n)} = \rho_{i_1} \otimes \rho_{i_2} \otimes \ldots\ldots \otimes \rho_{i_n}$$

HSW Theorem

$$C_p(\Phi) = \chi^*(\Phi) \quad \textit{Holevo Capacity}$$

$$= \max_{\{p_x, \rho_x\}} \min_{\sigma_B} D\left(\rho_{XB} \,\|\, \rho_X \otimes \sigma_B\right)$$

where

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \Phi(\rho_x);$$

$$\rho_X = \mathrm{Tr}_B \rho_{XB};$$

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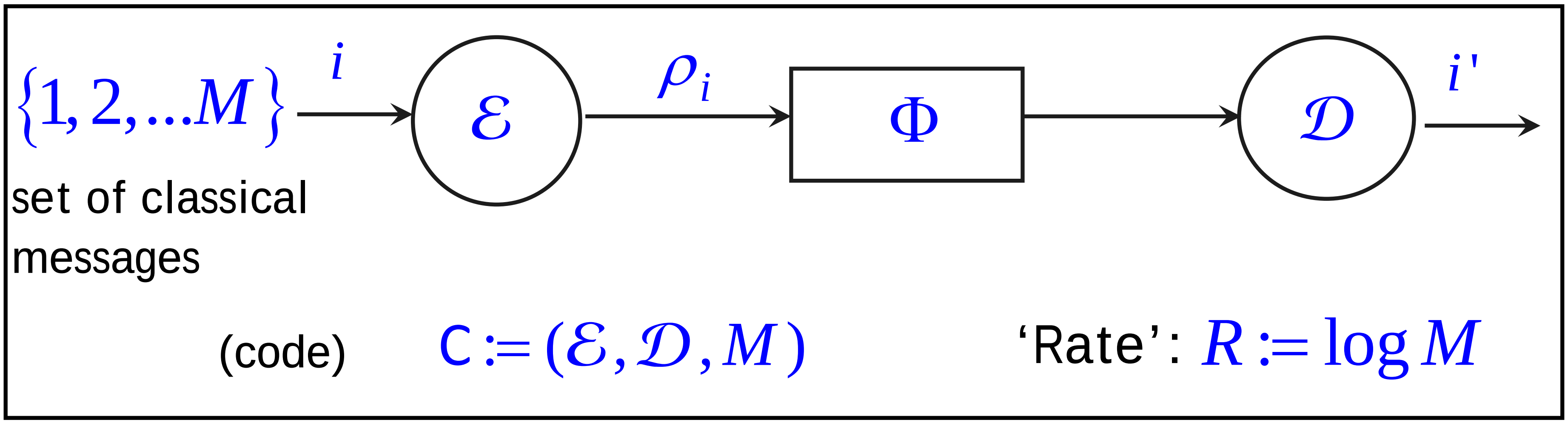
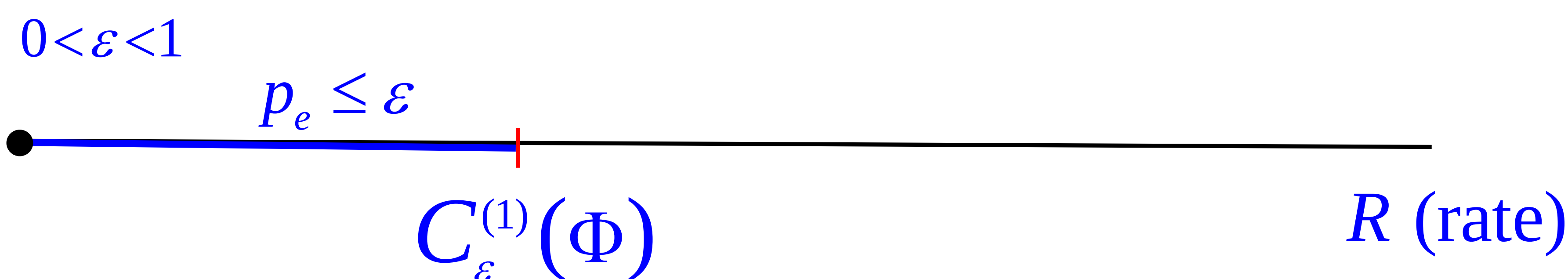
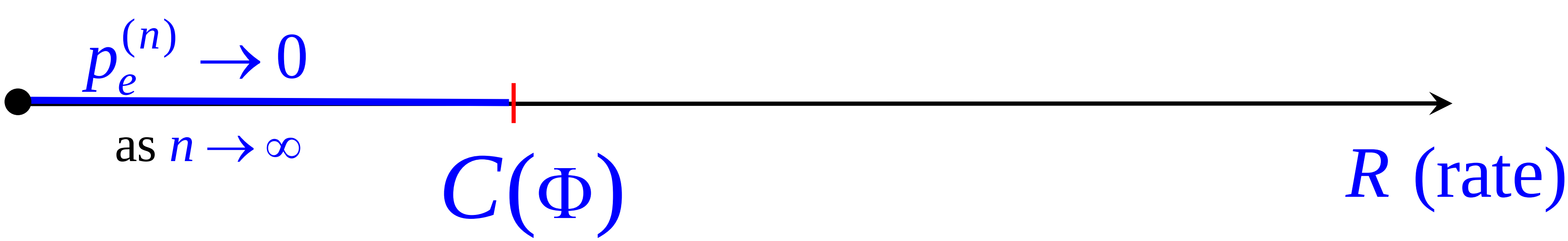
One-shot classical capacity





- Analogous to



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For any given value of the error tolerance threshold ε,

the transmission of classical info. through a noisy channel Φ

in the one-shot scenario, is characterized by

$$C_\varepsilon^{(1)}(\Phi) = \varepsilon - \textit{error one-shot classical capacity}$$

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- **Theorem 3:** $\forall\, 0<\varepsilon<1$

$$C_\varepsilon^{(1)}(\Phi) \approx \chi^*_{\max,\varepsilon'}(\Phi) = \max_{\{p_i,\rho_i\}} \min_{\sigma_B} D_{\max}^{\varepsilon'}\left(\rho_{XB} \,\|\, \rho_X \otimes \sigma_B\right)$$

smooth max-Holevo capacity

$\varepsilon' \propto (1-\varepsilon)$

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \Phi(\rho_x); \; \rho_X = \mathrm{Tr}_B \rho_{XB};$$

- compare with:

$$C_p(\Phi) = \chi^*(\Phi) = \max_{\{p_i,\rho_i\}} \min_{\sigma_B} D\left(\rho_{XB} \,\|\, \rho_X \otimes \sigma_B\right)$$

Holevo-capacity

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- **Theorem 3:** $\forall\, 0<\varepsilon<1$

$$C_\varepsilon^{(1)}(\Phi) \approx \chi^*_{\max,\varepsilon'}(\Phi) = \max_{\{p_i,\rho_i\}} \min_{\sigma_B} D^{\varepsilon'}_{\max}\left(\rho_{XB} \,\|\, \rho_X \otimes \sigma_B\right)$$

- Upper & lower bounds on $C_\varepsilon^{(1)}(\Phi)$ in terms of

smooth max-Holevo capacity $+$ $\varepsilon-$dependent terms

- In particular: $\forall\, 0\leq\varepsilon<1$

$$C_\varepsilon^{(1)}(\Phi) \leq \chi^*_{\max,\varepsilon'/4}(\Phi) + \log\frac{3}{\varepsilon'}$$

$\varepsilon' = (1-\varepsilon)$

(b) Binary hypothesis testing :one shot

- *null hypothesis* $H_0 : \rho$ *vs.* *alternative hypothesis* $H_1 : \sigma$

- POVM $\{T, I-T\}$
- Trade-off between the Type I and Type II error probabilities

$$\alpha(T) = \mathrm{Tr}((I-T)\rho) \qquad\qquad \beta(T) = \mathrm{Tr}(T\sigma)$$

- In the asymmetric setting of Stein's lemma:

$$\beta_\varepsilon(\rho,\sigma) := \min_T \{\beta(T) : \ \alpha(\mathrm{T}) \le \varepsilon\}; \qquad \varepsilon \in (0,1)$$

(b) Binary hypothesis testing : contd.

- In the asymptotic i.i.d. setting: Quantum Stein's lemma states

[Hiai & Petz; Hayashi & Nagaoka]

$$\lim_{n\to\infty}\left[-\frac{1}{n}\log\beta_\varepsilon(\rho^{\otimes n},\sigma^{\otimes n})\right] = D(\rho\|\sigma) \qquad \varepsilon\in(0,1)$$

- In the one-shot setting:

$$-\log\beta_\varepsilon(\rho,\sigma)\approx D_{\max}^{\varepsilon'}(\rho\|\sigma); \qquad \varepsilon'\propto(1-\varepsilon)$$

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From one-shot to the asymptotic i.i.d. setting

- *Since,*

$$\limsup_{n\to\infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \equiv D(\rho \| \sigma)$$

One-shot bounds → *asymptotic, i.i.d. result*

-- Quantum Stein's lemma.

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Summary

- *Max-relative entropy*
    - A generalized relative entropy
    - Acts as a parent for Renato's min-entropies
  - Its mathematical properties
  - Operational interpretations
    - *BSA of a bipartite state*
    - *multiple state discrimination*
    - *binary hypothesis testing*
- *Smooth max-relative entropy*
    - relation to other generalized relative entropies

      e.g. Relative Renyi entropy & sup-spectral divergence rate
- Its operational significances in one-shot information theory
    - *Transmission of classical info through a quantum channel*
    - *Type II error in hypothesis testing*

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Summary contd.

- *Max-relative entropy* ⟶ an entanglement monotone
  - Max-relative entropy of entanglement
    = log robustness
- Interesting operational interpretation in one-shot entanglement manipulation *[F.Brandao & ND]*

- Other occurrences of the (smooth) max-relative entropy
  - One-shot quantum state splitting *[M.Berta et al]*
  - Single-shot thermodynamics *[J.Oppenheim, M.Horodecki]*
  - Quantum state targeting *[T.Rudolph & R.Spekkens] ....*

References:

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- *M.Mosonyi & ND: Generalized relative entropies & the capacities of classical-quantum channels; J.Math. Phys 2009*

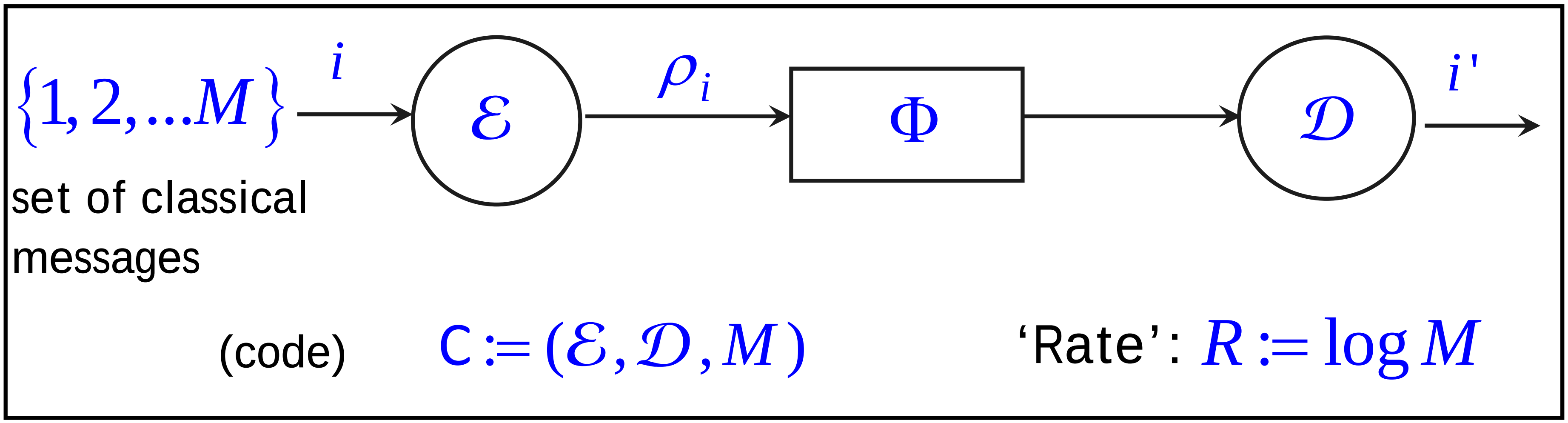
- *ND, M.Mosonyi, M-H.Hsieh, F.Brandao: Strong converses for classical information transmission & hypothesis testing ; arXiv 1106.3089*

- *ND & T.Rudolph : work in progress*

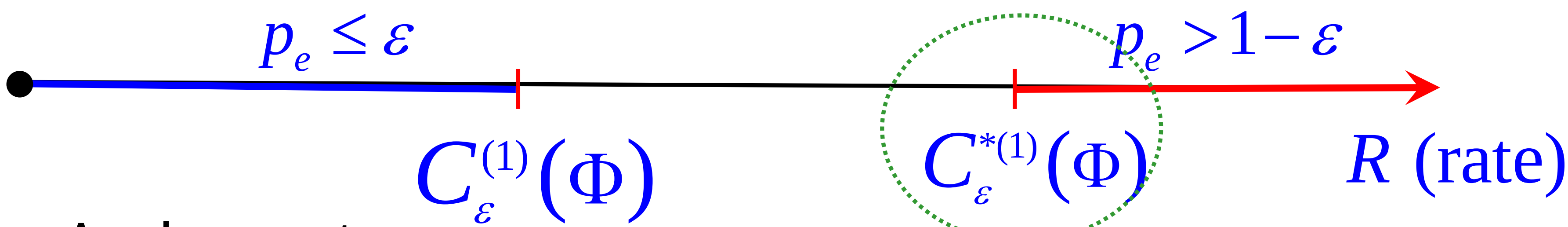
- *ND: Max-Relative Entropy alias Log Robustness; Intl. J.Q.Info. Thy, vol, 7, p.475, 2009.*

- *F. Brandao & ND: One-shot entanglement manipulation under non-entangling maps IEEE Trans. Inf. Thy, 2009*

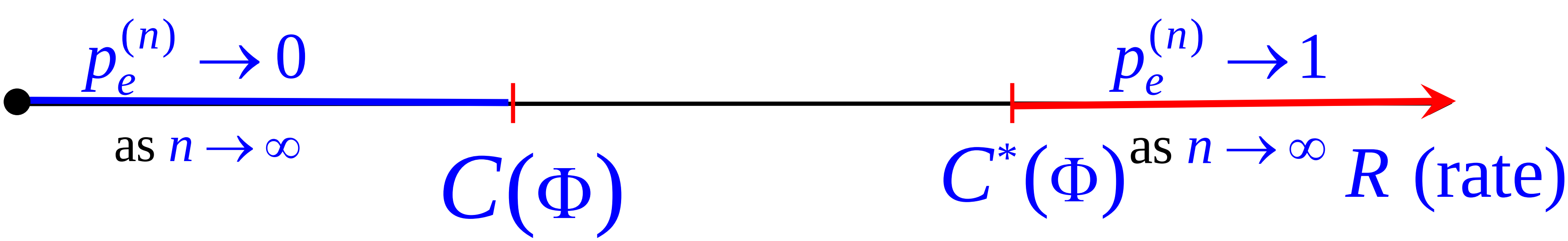
One-shot classical capacity and strong converse rate



$0<\varepsilon<1$



- Analogous to



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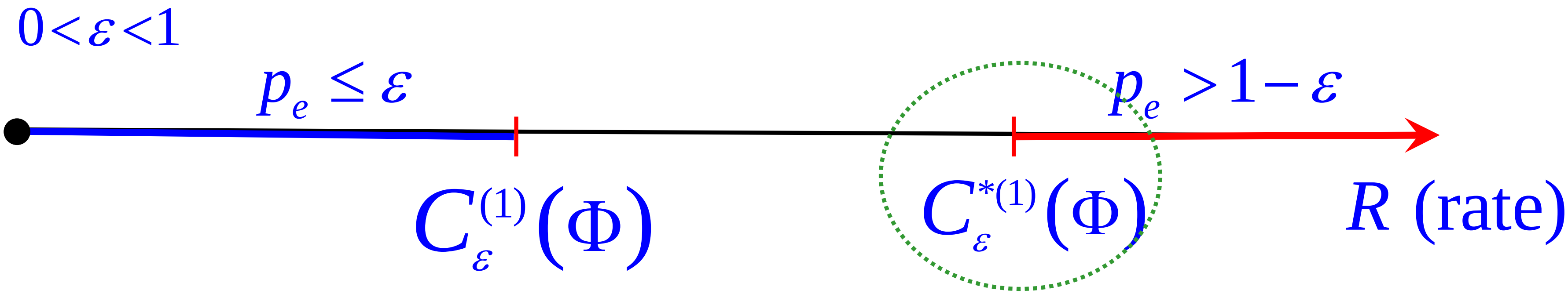

$$C_\varepsilon^{*(1)}(\Phi) := \inf\{\log M : \forall \mathsf{C} := (\mathcal{E}, \mathcal{D}, M),\, p_e > 1-\varepsilon\}$$

$$C_\varepsilon^{(1)}(\Phi) := \sup\{\log M : \exists \mathsf{C} := (\mathcal{E}, \mathcal{D}, M) \text{ s.t.} p_e \le \varepsilon\}$$

$$C_\varepsilon^{*(1)}(\Phi) = C_{1-\varepsilon}^{(1)}(\Phi)$$

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Transmission of classical info. through a noisy channel Φ

in the one-shot scenario, is characterized by a single quantity

for any given value of the error tolerance ε:

$$C_\varepsilon^{(1)}(\Phi) = \varepsilon - \textit{error one-shot classical capacity}$$

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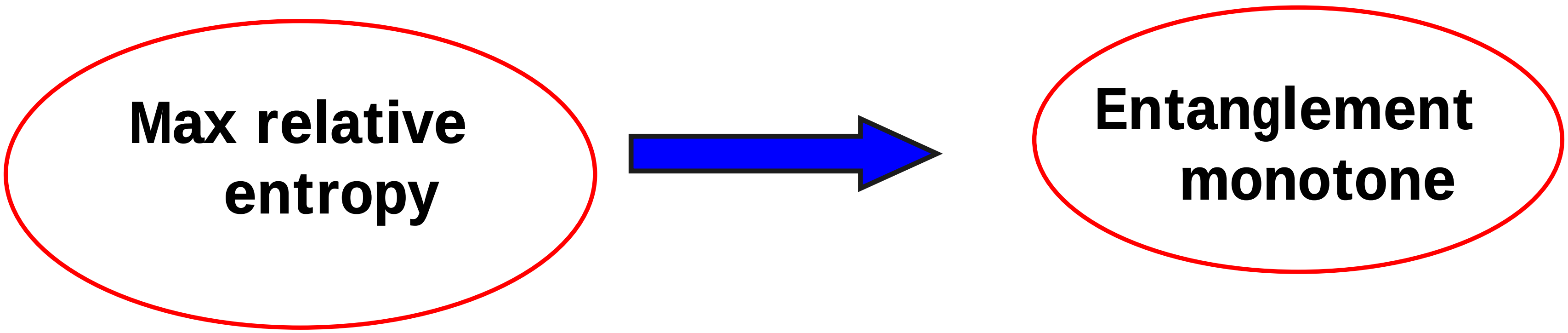

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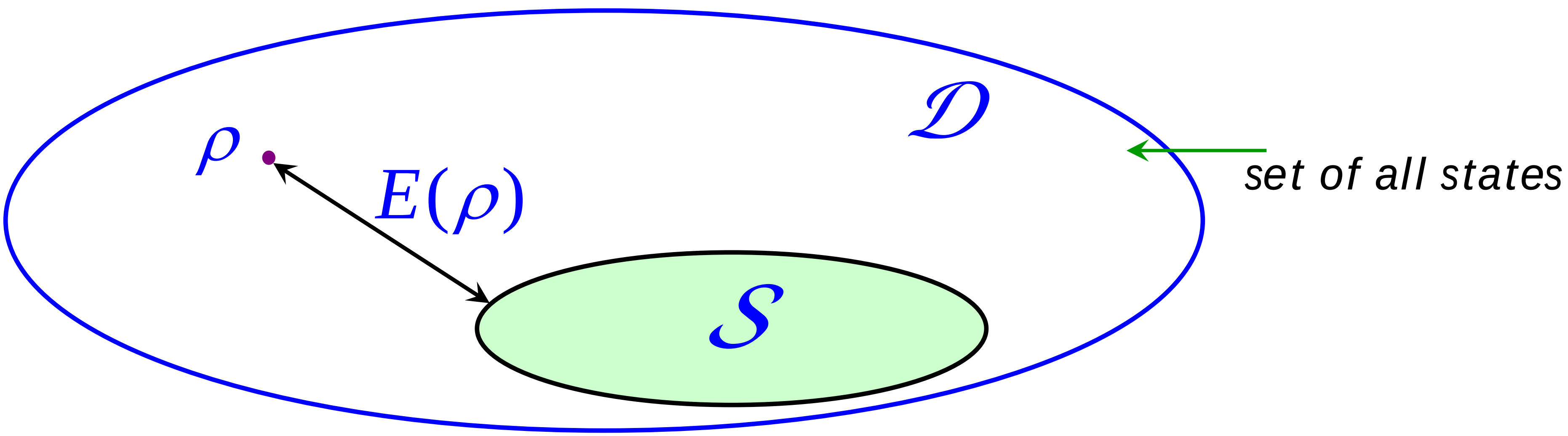
Entanglement monotones

- *Let* $\rho = \rho_{AB}$

> - $E(\rho)$ = *a measure of how entangled a state* ρ *is ;*
>
> *i.e., the amount of entanglement in the state* ρ

: "minimum distance" of ρ *from the set* $\mathcal{S}$ *of separable states.*



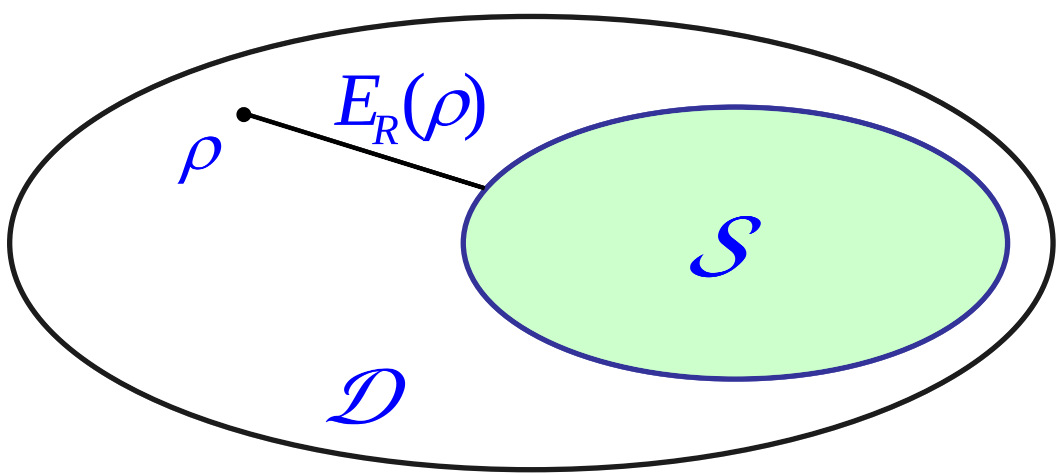
Relative Entropy of Entanglement

- One of the most important and fundamental entanglement measures for a bipartite state $\rho = \rho_{AB}$

$$E_R(\rho) := \min_{\sigma \in \mathcal{S}} D(\rho \| \sigma)$$

Quantum Relative Entropy

"distance"



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Entanglement Monotones

$$E_R(\rho) = \min_{\sigma \in \mathcal{S}} D(\rho \| \sigma)$$

relative entropy of entanglement

- We can define :

$$E_{\max}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\max}(\rho \| \sigma)$$

Max-relative entropy of entanglement

It can be proved to be an entanglement monotone!

- *Interesting operational interpretation in one-shot entanglement manipulation*

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Summary

- *Max-relative entropy*
  - A generalized relative entropy
  - Acts as a parent for Renato's min-entropies
- Its mathematical properties
- Operational interpretations
  - *BSA of a bipartite state*
  - *multiple state discrimination*
  - *binary hypothesis testing*
- *Smooth max-relative entropy*
  - relation to other generalized relative entropies

    e.g. Relative Renyi entropy & sup-spectral divergence rate
- Its operational significances in one-shot information theory
  - *Transmission of classical info through a quantum channel*
  - *Type II error in hypothesis testing*

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Summary contd.

- *Max-relative entropy* ⟶ an entanglement monotone
    - Max-relative entropy of entanglement

      = log robustness
- Interesting operational interpretation in one-shot entanglement manipulation *[F.Brandao & ND]*

- Other occurrences of the (smooth) max-relative entropy
    - One-shot quantum state splitting *[M.Berta et al]*
    - Single-shot thermodynamics *[J.Oppenheim, M.Horodecki]*
    - Quantum state targeting *[T.Rudolph & R.Spekkens] ….*

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Properties of $E_{\max}(\rho)$

- $E_{\max}(\rho) = 0$ *if* ρ *is separable*
- $E_{\max}(\Lambda_{\text{LOCC}}(\rho)) \leq E_{\max}(\rho)$ *weak monotonicity*

  (local operations & classical communication)
- $E_{\max}(\rho)$ is not changed by a local change of basis

etc.

- *In addition* $E_{\max}(\rho)$ *is a full entanglement monotone*

(does not increase on average under LOCC)

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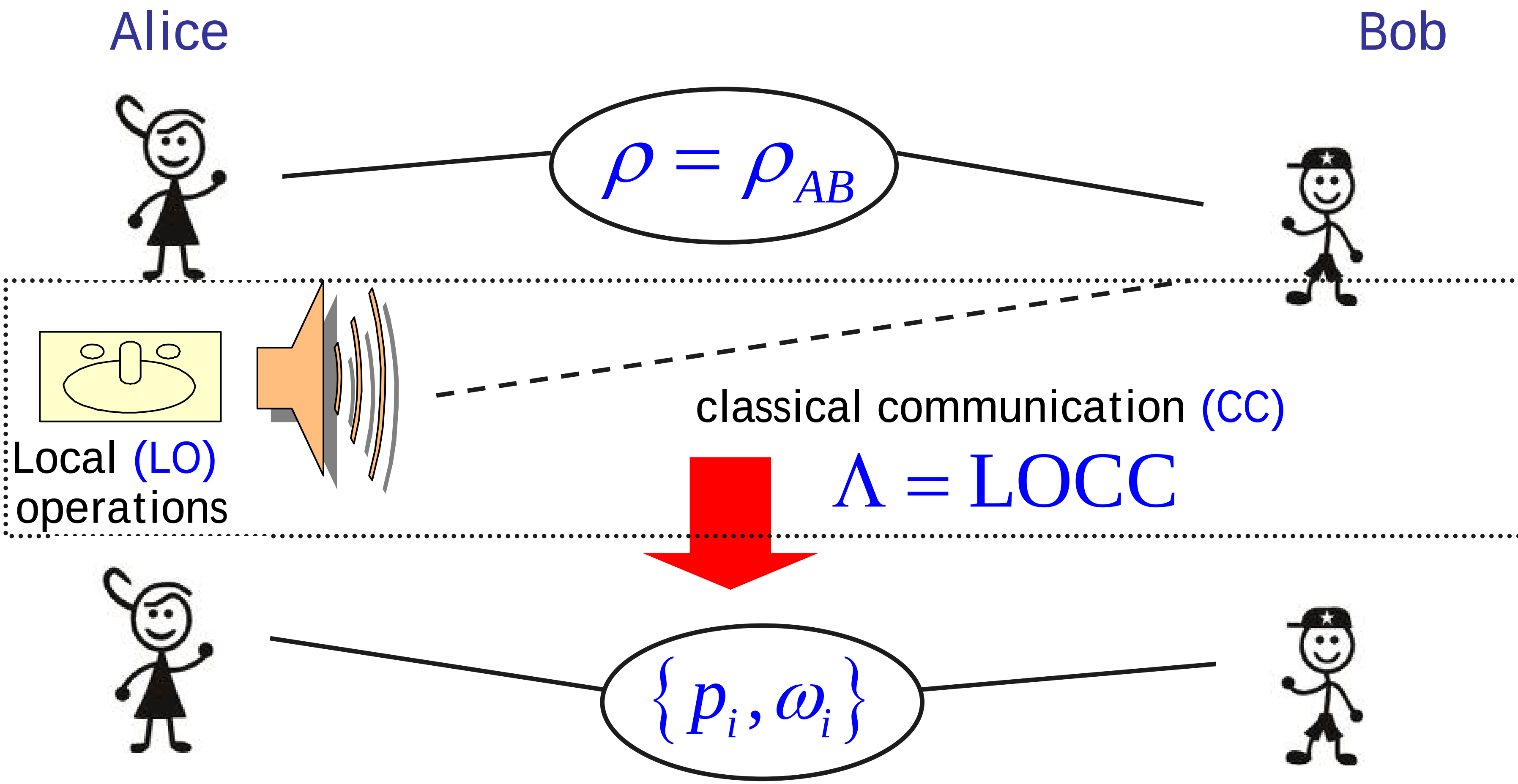

$\Lambda(\rho) = \omega_i$ with probability p_i

then $$E_{\max}(\rho) \geq \sum_i p_i E_{\max}(\omega_i)$$

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Relation to other entanglement monotones

$$E_{\max}(\rho) = LR_g(\rho)$$ = *log robustness* of ρ

-

$$LR_g(\rho) := \log\left(1 + R_g(\rho)\right)$$

$R_g(\rho)$ = *global robustness of* ρ

[Harrow & Nielsen]

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$$R_g(\rho) = \textit{global robustness}$$

[Harrow & Nielsen]

= a measure of the extent to which another state can be

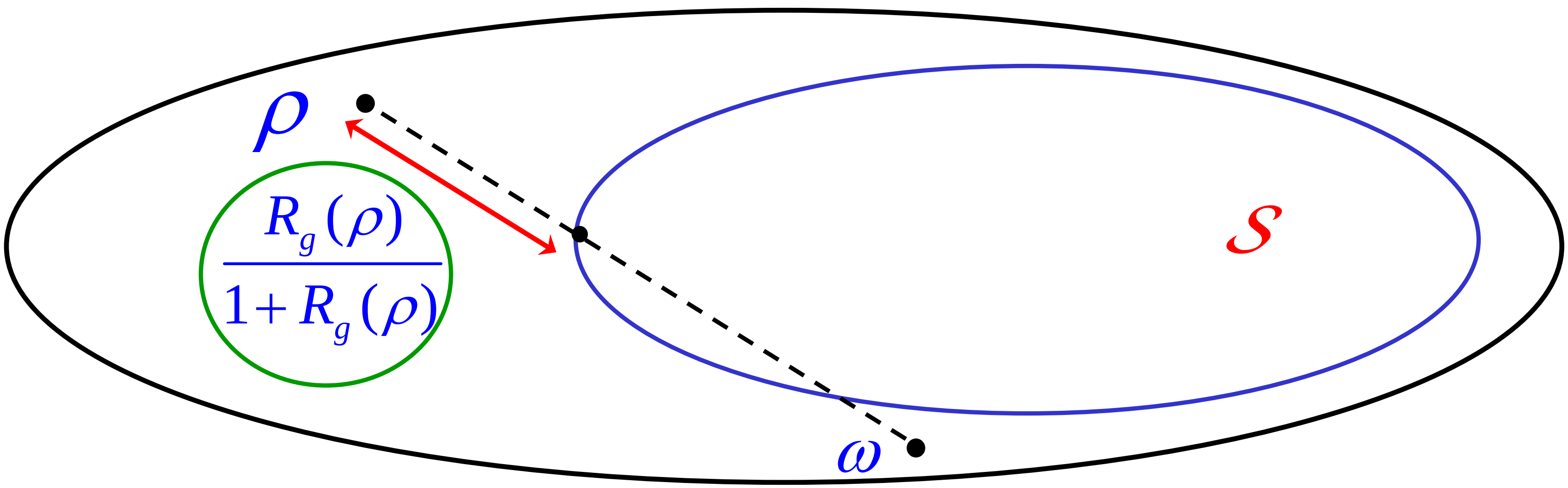
- mixed with ρ until the mixture becomes separable.

$$R_g(\rho) = \min \left\{ s : s \geq 0,\ \exists\ \omega \text{ s.t. } \left(\frac{1}{1+s}\rho + \frac{s}{1+s}\omega \right) \text{ is separable} \right\} \quad \textbf{.....(a)}$$

- A state ω for which the minimum is achieved in (a) is said to be the optimal state for $R_g(\rho)$

For an *optimal* ω $\left(\frac{1}{1+s}\rho + \frac{s}{1+s}\omega\right)$ is separable

for $s = R_g(\rho)$



- Further:

$R_g(\rho) = 0$	if	ρ	is separable
$R_g(\rho) = M - 1$	if	$\rho = \Psi_M$	*maximally entangled state*

$$E_{\max}(\rho) = LR_g(\rho) \; [= \log(1 + R_g(\rho))]$$

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$$E_{\max}(\rho) = LR_g(\rho) \; [= \log(1 + R_g(\rho))]$$

$$R_g(\rho) = \min \left\{ s : s \geq 0, \; \exists \, \omega \text{ s.t. } \left(\frac{1}{1+s}\rho + \frac{s}{1+s}\omega \right) \in \mathcal{S} \right\}$$

$$\rho + s\omega = (1+s)\sigma$$ where $\sigma \in \mathcal{S}$

$$\rho \leq (1+s)\sigma$$

$$\theta := (1+s)\sigma - \rho = s\omega$$

$$\mathrm{Tr}\,\theta := (1+s) - 1 = s$$

$$E_{\max}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\max}(\rho \,\|\, \sigma)$$

$$\log\left(\min\{ \lambda : \rho \leq \lambda\sigma \} \right)$$

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Thanks to:

Fernando Brandao, Francesco Buscemi, Min-Hsiu Hsieh,

Milan Mosonyi & Terry Rudolph