Towards a one-shot entanglement theory

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Part One: Introduction

Resource theory of (bipartite) entanglement

Entanglement is useful (quantum information processing) but expensive (difficult to establish and fragile to preserve)

☞ study of entanglement as a resource

- raw resources: bipartite quantum systems (in pure and/or mixed states)
- processing: local operations and classical communication (LOCC). (Why? Operational paradigm of "distant laboratories.")
- standard currency: the singlet state $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$. (Why? Perhaps because teleportation and superdense coding both use the singlet.)
- basic tasks: distillation (extraction of singlets from raw resources) and dilution (creation of generic bipartite states from singlets) by LOCC

Asymptotic manipulation of (bipartite) quantum correlations

$$\underbrace{\rho_{AB} \otimes \rho_{AB} \otimes \cdots \otimes \rho_{AB}}_{M_{\text{in}}} \xmapsto{\mathcal{L} \in \text{LOCC}} \underbrace{\sigma_{A'B'} \otimes \cdots \sigma_{A'B'}}_{N_{\text{out}}}$$

where A-systems belong to Alice, B-systems belong to Bob, and the transformation $\mathcal{L}$ is LOCC between Alice and Bob

- Jargon: M_{in} copies of the initial state ρ_{AB} are *diluted* into N_{out} copies of the target state $\sigma_{A'B'}$; equivalently, N_{out} copies of the target state $\sigma_{A'B'}$ are *distilled* from M_{in} copies of the initial state ρ_{AB}
- Task is optimized with respect to the resources created (*optimal distillation*, $N = N(M_{\text{in}})$) or those consumed (*optimal dilution*, $M = M(N_{\text{out}})$)
- Optimal rates are computed as $\lim_{M_{\text{in}} \to \infty} N(M_{\text{in}})/M_{\text{in}}$ (*optimal distillation rate*) and $\lim_{N_{\text{out}} \to \infty} M(N_{\text{out}})/N_{\text{out}}$ (*optimal dilution rate*)

Asymptotic entanglement distillation and dilution

- Entanglement distillation

$$\underbrace{\rho_{AB} \otimes \cdots \otimes \rho_{AB}}_{M_{\text{in}}} \xmapsto{\mathcal{L} \in \text{LOCC}} \underbrace{\Psi^-_{A'B'} \otimes \cdots \Psi^-_{A'B'}}_{N(M_{\text{in}})}$$

distillable entanglement: $E_D^\infty(\rho_{AB}) = \lim_{M_{\text{in}} \to \infty} N(M_{\text{in}})/M_{\text{in}}$

- Entanglement dilution

$$\underbrace{\Psi^-_{AB} \otimes \cdots \otimes \Psi^-_{AB}}_{M(N_{\text{out}})} \xmapsto{\mathcal{L} \in \text{LOCC}} \underbrace{\sigma_{A'B'} \otimes \cdots \sigma_{A'B'}}_{N_{\text{out}}}$$

entanglement cost: $E_C^\infty(\sigma_{A'B'}) = \lim_{N_{\text{out}} \to \infty} M(N_{\text{out}})/N_{\text{out}}$

Criticisms to this approach

The asymptotic framework is *operational* but not *practical*, for two reasons:

- asymptotic achievability (and often without knowing how fast the limit is approached)
- i.i.d. assumption: hardly satisfied in practical scenarios

A third remark: the asymptotic i.i.d. argument mixes information theory and probability theory. As noticed by Han and Verdú, we'd like to distinguish what is information theory from what is probability theory.

The one-shot case

- One-shot entanglement distillation:

$$\rho_{AB} \xmapsto{\mathcal{L}\in\text{LOCC}} \underbrace{\Psi^-_{A'B'} \otimes \cdots \Psi^-_{A'B'}}_{N_{\max}(\rho_{AB})}.$$

- One-shot entanglement dilution:

$$\underbrace{\Psi^-_{AB} \otimes \cdots \otimes \Psi^-_{AB}}_{M_{\min}(\sigma_{A'B'})} \xmapsto{\mathcal{L}\in\text{LOCC}} \sigma_{A'B'}.$$

- Correspondingly,
  - one-shot distillable entanglement: $E_D^{(1)}(\rho_{AB}) = N_{\max}(\rho_{AB})$;
  - one-shot entanglement cost: $E_C^{(1)}(\sigma_{A'B'}) = M_{\min}(\sigma_{A'B'})$

Allowing for finite accuracy

Again, with an eye to practical implementations:

One-shot entanglement ε-distillation:

$$\rho_{AB} \xmapsto{\mathcal{L}\in\text{LOCC}} \tilde{\rho}_{A'B'} \overset{\varepsilon}{\approx} \underbrace{\Psi^-_{A'B'} \otimes \cdots \Psi^-_{A'B'}}_{N_{\max}(\rho_{AB};\varepsilon)}.$$

One-shot entanglement ε-dilution

$$\underbrace{\Psi^-_{AB} \otimes \cdots \otimes \Psi^-_{AB}}_{M_{\min}(\sigma_{A'B'};\varepsilon)} \xmapsto{\mathcal{L}\in\text{LOCC}} \tilde{\sigma}_{A'B'} \overset{\varepsilon}{\approx} \sigma_{A'B'}.$$

Correspondingly,

- one-shot ε-distillable entanglement: $E_D^{(1)}(\rho_{AB};\varepsilon) = N_{\max}(\rho_{AB};\varepsilon)$;
- one-shot entanglement ε-cost: $E_C^{(1)}(\sigma_{A'B'};\varepsilon) = M_{\min}(\sigma_{A'B'};\varepsilon)$

Outline of the talk

- one-shot distillable entanglement (pure state case)
- generalized entropies: $S_{\min}$ and $S_{\max}$
- one-shot entanglement cost (pure state case)
- overview of the mixed state case: asymptotic results
- relative Rényi entropies and derived quantities
- mixed state case: one-shot results
- comparison and discussion

Part Two:
The Strange Case of Pure States

Case study: pure bipartite states

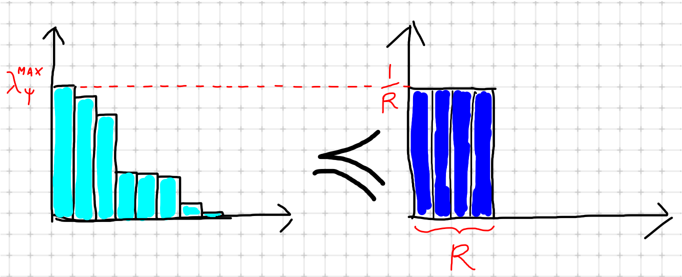
$$|\psi_{AB}\rangle \xmapsto{\mathcal{L}\in\text{LOCC}} |\phi_{A'B'}\rangle$$

True in this case (but grossly false in general):

- all the properties of a pure bipartite state ψ_{AB} are determined by the list of eigenvalues $\vec{\lambda}_\psi$ of the reduced density matrix $\psi_A = \mathrm{Tr}_B[\psi_{AB}]$;
- Lo and Popescu: the action of a general LOCC map on a pure state can be also obtained by another one-way, one-round LOCC map;
- Nielsen: there exists an LOCC transformation mapping ψ_{AB} into $\phi_{A'B'}$ if and only if $\psi_A \prec \phi_{A'}$, i.e., $\sum_{i=1}^k \lambda_\psi^{\downarrow i} \leqslant \sum_{i=1}^k \lambda_\phi^{\downarrow i}$, for all k;
- asymptotic reversibility (total ordering): $E_D^\infty(\psi_{AB}) = E_C^\infty(\psi_{AB}) = S(\psi_A)$.

One-shot zero-error distillable entanglement: $E_D^{(1)}(\psi_{AB};0)$

Nielsen: given an initial pure state ψ_{AB}, a maximally entangled state of rank R, i.e. $R^{-1/2}\sum_{i=1}^{R}|i\rangle|i\rangle$, can be distilled if and only if $\lambda_\psi^{\max}\equiv\lambda_\psi^{\downarrow 1}\leqslant R^{-1}$, $\lambda_\psi^{\downarrow 1}+\lambda_\psi^{\downarrow 2}\leqslant 2R^{-1}$, and so on.

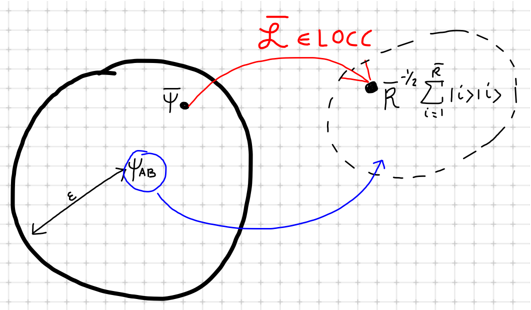


☞ A maximally entangled state of rank $R=\left\lfloor\frac{1}{\lambda_\psi^{\max}}\right\rfloor$ can always be distilled *exactly*, i.e.,

$$E_D^{(1)}(\psi_{AB};0)\geqslant\log_2\left\lfloor\frac{1}{\lambda_\psi^{\max}}\right\rfloor .$$

Finite accuracy: $E_D^{(1)}(\psi_{AB};\varepsilon)$

Consider the set of pure states $B^*_\varepsilon(\psi_{AB}) := \left\{ |\bar{\psi}_{AB}\rangle : \bar{\psi}_{AB} \overset{\varepsilon}{\approx} \psi_{AB} \right\}$



☞ A maximally entangled state of rank $\bar{R} = \left\lfloor \frac{1}{\lambda^{\max}_{\bar{\psi}}} \right\rfloor$ can always be distilled up to an ε-error, i.e.,

$$E_D^{(1)}(\psi_{AB};\varepsilon) \geqslant \max_{\bar{\psi}\in B^*_\varepsilon(\psi)} \log_2 \left\lfloor \frac{1}{\lambda^{\max}_{\bar{\psi}}} \right\rfloor .$$

Getting the right smoothing

With $B^*_\varepsilon(\psi_{AB}) := \left\{|\bar{\psi}_{AB}\rangle : \bar{\psi}_{AB} \stackrel{\varepsilon}{\approx} \psi_{AB}\right\}$:

$$E_D^{(1)}(\psi_{AB};\varepsilon) \geqslant \underbrace{\max_{\bar{\psi}\in B^*_\varepsilon(\psi)} \log_2 \left\lfloor \frac{1}{\lambda^{\max}_{\bar{\psi}}} \right\rfloor}_{\mathrm{f}(\psi_{AB},\varepsilon)} \equiv S^\varepsilon_{\min}(\psi_A)$$

Given a (mixed) state ρ, define the set of (mixed) states $B_\varepsilon(\rho) := \left\{\bar{\rho} : \bar{\rho} \stackrel{\varepsilon}{\approx} \rho\right\}$. The *smoothed min-entropy* of ρ is defined as (Renner) $S^\varepsilon_{\min}(\rho) := \max_{\bar{\rho}\in B_\varepsilon(\rho)} [-\log_2 \lambda_{\max}(\bar{\rho})]$.

$S_{\min}$ is the one-shot distillable entanglement

A converse also holds:

$$S^{\varepsilon}_{\min}(\psi_A) \leqslant E^{(1)}_D(\psi_{AB};\varepsilon) \leqslant S^{\varepsilon'}_{\min}(\psi_A) - \log_2(1-2\sqrt{\varepsilon}). \qquad \left[\varepsilon' = 2^{\frac{5}{4}}\varepsilon^{\frac{1}{8}}\right]$$

☞ min-entropy of the reduced state ≈ one-shot distillable entanglement of a pure bipartite state.

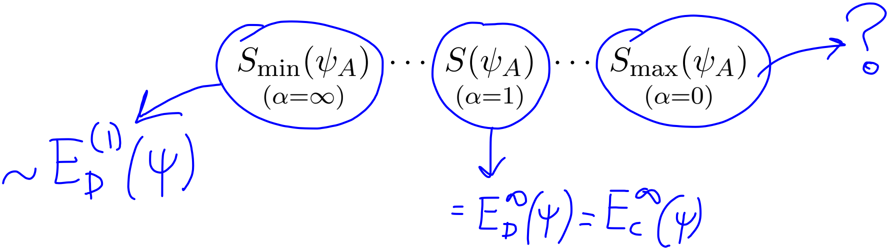


Figure: is $S_{\max}$ associated with anything?

$S_{\max}$ is the one-shot entanglement cost

- Vidal, Jonathan, and Nielsen: a pure bipartite state ψ_{AB} can be obtained by LOCC from a maximally entangled state of rank R with a minimum error of $\varepsilon = 1 - \sum_{i=1}^{R} \lambda_{\psi}^{\downarrow i}$.
- As a consequence, $E_C^{(1)}(\psi_{AB}; 0) = \log_2 \operatorname{rank} \psi_A = S_{\max}(\psi_A)$.
- With finite accuracy:

$$E_C^{(1)}(\psi_{AB}; \varepsilon) \simeq S_{\max}^{\varepsilon}(\psi_A),$$

where $S_{\max}^{\varepsilon}(\rho) = \min_{\bar{\rho} \in B_{\varepsilon}(\rho)} S_{\max}(\bar{\rho})$.

Summary of the pure state case

$$\boxed{\begin{array}{ccccccc} E_D^{(1)}(\psi_{AB};\varepsilon) & \simeq & S^{\varepsilon}_{\min}(\psi_A) & \leqslant & S^{\varepsilon}_{\max}(\psi_A) & \simeq & E_C^{(1)}(\psi_{AB};\varepsilon) \\ \downarrow & & \searrow & & \swarrow & & \downarrow \\ E_D^{\infty}(\psi_{AB}) & = & & S(\psi_A) & & = & E_C^{\infty}(\psi_{AB}) \end{array}}$$

where "$F(\rho;\varepsilon) \to G(\rho)$" means $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}F(\rho^{\otimes n};\varepsilon) = G(\rho)$

☞ asymptotic reversibility holds for pure states

One-shot irreversibility gap for pure states

Reversibility only holds asymptotically. Define the one-shot irreversibility gap as

$$\begin{aligned}\Delta(\psi_{AB};\varepsilon) :&= E_C^{(1)}(\psi_{AB};\varepsilon) - E_D^{(1)}(\psi_{AB};\varepsilon)\\ &\simeq S^{\varepsilon}_{\max}(\psi_A) - S^{\varepsilon}_{\min}(\psi_A)\end{aligned}$$

This quantity is related with the communication cost C of transforming an initial pure state ψ^i_{AB} into a final state $\psi^f_{A'B'}$ (Hayden and Winter, 2003):

$$2C \geqslant \Delta(\psi^f_{A'B'};0) - \Delta(\psi^i_{AB};0).$$

☞ "Increasing irreversibility requires communication."

Part Three: The Complicated Case of Mixed States (an overview)

Mixed state case: asymptotic i.i.d. results

Distillable entanglement and entanglement cost are naturally quantified by *different functions* of ρ_{AB} (Hayden, Horodecki, Terhal, 2001; Devetak, Winter, 2005):

$$\begin{array}{ccccc} E_D^\infty(\rho_{AB}) & & & & E_C^\infty(\rho_{AB}) \\ \wr & & & & \wr \\ I_c^{A\to B}(\rho_{AB}) & \xhookrightarrow{\text{pure states}} & S(\rho_A) & \xhookleftarrow{\text{pure states}} & \min_{\mathfrak{E}} \sum_i p_i S(\psi_A^i) \end{array}$$

where:

- $I_c^{A\to B}(\rho_{AB}) = S(\rho_B) - S(\rho_{AB}) = -H(\rho_{AB}|B)$: coherent information
- $\min_{\mathfrak{E}} \sum_i p_i S(\psi_A^i)$ is done over all pure-state ensemble decompositions $\rho_{AB} = \sum_i p_i \psi_{AB}^i$: entanglement of formation $E_F(\rho_{AB})$

Relative entropies and derived quantities

All such entropic quantities are originated from a common parent

Relative entropy:

$$S(\rho\|\sigma) = \mathrm{Tr}\left[\rho\log_2\rho - \rho\log_2\sigma\right]$$

1. $S(\rho) := -\mathrm{Tr}[\rho\log_2\rho] = -S(\rho\|\mathbb{1})$
2. $H(\rho_{AB}|B) := S(\rho_{AB}) - S(\rho_B) = -\min_{\sigma_B} S(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B)$
3. $I_c^{A\to B}(\rho_{AB}) := -H(\rho_{AB}|B)$

Relative Rényi entropy of order zero:

$$S_0(\rho\|\sigma) = -\log_2\mathrm{Tr}\left[\Pi_\rho\,\sigma\right]$$

1. $S_0(\rho) := -S_0(\rho\|\mathbb{1}) = S_{\max}(\rho)$
2. $H_0(\rho_{AB}|B) := -\min_{\sigma_B} S_0(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B)$
3. $I_0^{A\to B}(\rho_{AB}) := -H_0(\rho_{AB}|B)$

Technical remark: quasi-entropies

In our proofs we employed the notion of quasi-entropies (Petz, 1986)

$$S_\alpha^P(\rho\|\sigma) = \frac{1}{\alpha-1}\log_2 \mathrm{Tr}\left[\sqrt{P}\rho^\alpha\sqrt{P}\ \sigma^{1-\alpha}\right],$$

defined for $\rho, \sigma \geqslant 0$, $0 \leqslant P \leqslant \mathbb{1}$, and $\alpha \in (0,\infty)/\{1\}$.

In particular, we enjoyed working with

$$S_0^P(\rho\|\sigma) = \lim_{\alpha\searrow 0} S_\alpha^P(\rho\|\sigma) = -\log_2 \mathrm{Tr}\left[\sqrt{P}\Pi_\rho\sqrt{P}\ \sigma\right],$$

smoothing w.r.t. ρ or P, depending on the problem at hand.

Mixed state case: one-shot results

Keeping in ming the asymptotic i.i.d. case:

$$\begin{array}{ccccc} E_D^{\infty}(\rho_{AB}) & & & & E_C^{\infty}(\rho_{AB}) \\ \wr & & & & \wr \\ I_c^{A\to B}(\rho_{AB}) & \xhookrightarrow{\text{pure}} & S(\rho_A) & \xhookleftarrow{\text{pure}} & \min_{\mathfrak{E}} \sum_i p_i S(\psi_A^i) \\ & & & & \| \\ & & & & \min_{\mathfrak{E}} H(\rho_{RA}|R) \end{array}$$

Here are the one-shot analogues:

$$\begin{array}{ccccccc} E_D^{(1)}(\rho_{AB};\varepsilon) & & & & & & E_C^{(1)}(\rho_{AB};\varepsilon) \\ \wr & & & & & & \wr \\ I_{0,\varepsilon}^{A\to B}(\rho_{AB}) & \xhookrightarrow{\text{pure}} & S_{\text{min}}^{\varepsilon}(\rho_A) & \leqslant & S_{\text{max}}^{\varepsilon}(\rho_A) & \xhookleftarrow{\text{pure}} & \min_{\mathfrak{E}} H_0^{\varepsilon}(\rho_{RA}|R) \end{array}$$

where $\min_{\mathfrak{E}} H_0^{\varepsilon}(\rho_{RA}|R)$ is done over all cq-extensions $\rho_{RAB} = \sum_i p_i |i\rangle\langle i|_R \otimes \psi_{AB}^i$, such that $\mathrm{Tr}_R[\rho_{RAB}] = \rho_{AB}$

A by-product worth noticing

Since $E_C^\infty(\rho_{AB}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}E_C^{(1)}(\rho_{AB}^{\otimes n};\varepsilon)$, from the previous slide:

$$\min_{\mathfrak{E}} H_0^\varepsilon(\rho_{RA}|R) \xrightarrow{\lim_{\varepsilon\to 0}\frac{1}{n}\lim_{n\to\infty}} \min_{\mathfrak{E}} H(\rho_{RA}|R)$$

Both well-known guests of the zoo of entanglement measures:

- $\min_{\mathfrak{E}} H(\rho_{RA}|R)$ is the *entanglement of formation* (Bennett et al, 1996) $E_F(\rho_{AB}) = \min_{\mathfrak{E}}\sum_i p_i S(\psi_A^i)$
- $\min_{\mathfrak{E}} H_0(\rho_{RA}|R)$ is the logarithm of the *generalized Schmidt rank* (Terhal, Horodecki, 2000) $E_{sr}(\rho_{AB}) = \log_2\min_{\mathfrak{E}}\max_i \operatorname{rank}\psi_A^i$

By introducing a smoothed Schmidt rank as follows:

$$E_{sr}^\varepsilon(\rho_{AB}) := \min_{\bar\rho_{AB}\in B_\varepsilon(\rho_{AB})} E_{sr}(\bar\rho_{AB}),$$

we have implicitly proved that

$$\boxed{\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}E_{sr}^\varepsilon(\rho_{AB}^{\otimes n}) = \lim_{n\to\infty}\frac{1}{n}E_F(\rho_{AB}^{\otimes n}).}$$

Conclusions and open questions

- mix- and max-entropies naturally arise also in one-shot entanglement theory
- pleasant formal analogy with the asymptotic i.i.d. case: just replace $S(\rho\|\sigma)$ by $S_0(\rho\|\sigma)$ (but, first, find the right expression to replace!)
- sometimes, the one-shot analysis uncovers new relations between known functions (e.g. the regularized entanglement of formation equals the smoothed-and-regularized log-Schmidt rank)
- increasing irreversibility requires communication: what about mixed states?
- other operational paradigms: SEPP done (Fernando and Nila); what about LOSR?
- one-shot squashed entanglement: one-shot quantum conditional mutual information?

La Fine.