Aim: to estimate the edges of a graphical model. For this, we require:

### Definitions

- **Prior**
  - A prior probability distribution $\Theta$ over the set $G$ of decomposable graphs on a vertex set $V$. This is difficult to work with:
    - Number of undirected graphs is $2^{\binom{n}{2}}$, $g^{2^\binom{n}{2}}$.
    - Number of decomposable graphs is difficult to determine (NP-hard problem [4]).

- **Definition**
  - $\Theta$ has the structural Markov property if for any $A, B \subseteq V$:
    - $G_0 \subseteq G_0(A, B)$ is a decomposition of $G$ [9].
    - Separated components have independent structure.

- **Theorem**
  - A positive distribution $\Theta$ over the set of decomposable graphs is structurally Markov if and only if it is a member of the exponential family:
    - $\Theta(G, \omega) \propto \exp(\omega \cdot t(G))$, $\omega \in \mathbb{R}^{2^{|V|}}$.
    - The sufficient statistic $t(G) = \{c_0 \cup c_v \in \mathbb{Z}^{|V|} : c_0 \cap c_v = \emptyset\}$.
    - If $A$ is a clique, $c_A = 1 - \text{(multiplicity of } A)$. Otherwise, $c_A = 0$.
    - The structural Markov property characterises this exponential family.

### Computation

We require a method of computing the probability of each graph under the posterior distribution. By (1) and (2), we know that it will be a member of the exponential family.

- **Theorems**
  - If the prior distribution over the set of decomposable graphs is $\Theta(\cdot ; \omega)$, and the distributions of $X|G$ are compatible, then the posterior distribution of $G|X = x$ is:
    - $\Theta(\cdot; \omega + [\log p(x|G)]_{c_v})$.
    - The above exponential family is conjugate under sampling from compatible models.
    - We need “only” need a parameter of dimension $\sim 2^{|V|}$ instead of $\sim 2^{|V|^3}$.

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### Bibliography