Some results concerning maximum Rényi entropy distributions

Oliver Johnson    Christophe Vignat

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English abstract: We consider the Student-\(t\) and Student-\(r\) distributions, which maximise Rényi entropy under a covariance condition. We show that they have information-theoretic properties which mirror those of the Gaussian distributions, which maximise Shannon entropy under the same condition. We introduce a convolution which preserves the Rényi maximising family, and show that the Rényi maximisers are the case of equality in a version of the Entropy Power Inequality. Further, we show that the Rényi maximisers satisfy a version of the heat equation, motivating the definition of a generalized Fisher information.

French title: Quelques résultats au sujet des distributions à entropie de Rényi maximale.

French abstract: Nous considérons les distributions de types Student-\(t\) et Student-\(r\) qui maximisent l’entropie de Rényi sous contrainte de covariance. Nous montrons qu’elles possèdent des propriétés informationnelles similaires à celles des distributions Gaussiennes, lesquelles maximisent l’entropie de Shannon sous la même contrainte. Nous montrons que ces distributions sont stables pour un certain type de convolution et qu’elles saturent une inégalité de la puissance entropicque. De plus nous montrons que les lois à entropie de Rényi maximale vérifient une équation de la chaleur, ce qui permet de définir une information de Fisher généralisée.

Keywords: Entropy Power Inequality, Fisher information, heat equation, maximum entropy, Rényi entropy

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1 Introduction

It is natural to ask whether the Shannon entropy of a $n$-dimensional random vector with density $p$, defined as

$$H(p) = -\int p(x) \log p(x) dx,$$

represents the only possible measure of uncertainty. For example, Rényi [11] introduces axioms on how we would expect such a measure to behave, and shows that these axioms are satisfied by a more general definition is possible, as follows:

**Definition 1.1** Given a probability density $p$ valued on $\mathbb{R}^n$, for $q \neq 1$ define the $q$-Rényi entropy to be:

$$H_q(p) = \frac{1}{1-q} \log \left( \int p(x)^q dx \right).$$

Note that by L'Hôpital’s rule, since $\frac{d}{dt} a^t = a^t \log a$,

$$\lim_{q \to 1} H_q(p) = \lim_{q \to 1} \frac{-\int p(x)^q \log p(x) dx}{\int p(x)^q dx} = H(p). \quad (1)$$

As Gnedenko and Korolev [8] remark, under a variety of natural conditions the distributions which maximise Shannon entropy are well-known ones, with interesting properties. This paper gives parallels to some of these properties for the Rényi maximisers.

1. Under a covariance constraint Shannon entropy is maximised by the Gaussian distribution. In Proposition 2.4 we review the fact that under a covariance constraint Rényi entropy is maximised by Student distributions.

2. The Gaussians have the appealing property of stability (that is, given $Z_1$ and $Z_2$ Gaussians, $Z_1 + Z_2$ is also Gaussian). In Definition 3.2, we introduce the $\ast$-convolution, which generalizes the addition operation. In Lemma 3.3, we extend the stability property by showing that if $R_1$ and $R_2$ are Rényi maximisers then so is $R_1 \ast R_2$. 

2
3. The Entropy Power Inequality (see Equation (8) below) shows that the Gaussian represents the extreme case for how much entropy can change on addition. Theorem 3.4 gives the equivalent of an Entropy Power Inequality, with the Rényi maximisers playing an extremal role.

4. The Gaussian density satisfies the heat equation, which leads to a representation of Shannon entropy as an integral of Fisher Informations (known as the de Bruijn identity). In Theorem 4.1 we show that the Rényi densities satisfy a generalization of the heat equation, and deduce what quantity must replace the Fisher information in general.

2 Definitions of Rényi entropy maximisers

As in Costa, Hero and Vignat [4], we can identify the Rényi maximising densities, which are Student-\(t\) and Student-\(r\) distributions, with the following definitions.

**Definition 2.1** For \(\frac{n}{n+2} < q < 1\) and \(q \neq 1\), define the \(n\)-dimensional probability density \(g_{q,C}(x)\) as

\[
g_{q,C}(x) = A_q \left(1 - (q-1)\beta x^T C^{-1} x\right)^{\frac{1}{q-1}}
\]

with

\[
\beta = \beta_q = \frac{1}{2q - n \left(1 - q\right)},
\]

and normalization constants

\[
A_q = \begin{cases} 
\frac{\Gamma \left(\frac{1}{1-q}\right) \left(\beta(1-q)\right)^{n/2}}{\Gamma \left(\frac{1}{1-q} - \frac{n}{2}\right) \pi^{n/2} |C|^{1/2}} & \text{if } \frac{n}{n+2} < q < 1, \\
\frac{\Gamma \left(\frac{q}{q-1} + \frac{n}{2}\right) \left(\beta(q-1)\right)^{n/2}}{\Gamma \left(\frac{q}{q-1}\right) \pi^{n/2} |C|^{1/2}} & \text{if } q > 1.
\end{cases}
\]

Here \(x_+ = \max(x, 0)\) denotes the positive part. We write \(R_{q,C}\) for a random variable with density \(g_{q,C}\), which has mean \(0\) and covariance \(C\).

Notice that if we write \(\Omega_{q,C}\) for the support of \(g_{q,C}\), then for \(q > 1\), \(\Omega_{q,C} = \{x : x^T C^{-1} x \leq 2q/(q - 1) + n\}\), and for \(q < 1\), \(\Omega_{q,C} = \mathbb{R}^n\).

Note further that since \(\lim_{q \to 1} \Gamma(1/(1-q))(1-q)^{n/2}/\Gamma(1/(1-q) - n/2) = 1\) and \(\lim_{q \to 1} (1 - (q-1)\beta x^T C^{-1} x)^{\frac{1}{q-1}} = \exp(-x^T C^{-1} x/2)\), the limit
\[ \lim_{q \to 1} g_{q,C}(x) = g_{1,C}(x) = ((2\pi)^n |C|)^{-1/2} \exp(-x^T C^{-1} x/2), \] the Gaussian density. Throughout this paper, we write \( Z_{C} \) for a \( N(0,C) \) random variable.

We first show that \( g_{q,C} \) are the Rényi entropy maximisers, using Lemma 1 of Lutwak, Yang and Zhang [10], which extends the classical Gibbs inequality.

**Definition 2.2** For \( q \neq 1 \), given \( n \)-dimensional probability densities \( f \) and \( g \), define the relative \( q \)-Rényi entropy distance from \( f \) to \( g \) to be

\[
D_q(f\|g) = \frac{1}{1-q} \log \left( \int g^{q-1}(x) f(x) dx \right) + \frac{1-q}{q} H_q(g) - \frac{1}{q} H_q(f).
\]

For \( q = 1 \), we write \( D_1(f\|g) = \int f(x) \log(f(x)/g(x)) dx \) for the standard relative entropy. We justify this as an extension by continuity; as \( q \to 1 \), as in (1), \( D_q(f\|g) \to -\int f(x) \log g(x) dx - H_1(f) = D_1(f\|g) \).

**Lemma 2.3** For any \( q > 0 \), and for any probability densities \( f \) and \( g \), the relative entropy \( D_q(f\|g) \geq 0 \), with equality if and only if \( f = g \) almost everywhere.

**Proof** The case \( q = 1 \) is well-known. For \( q \neq 1 \), as in Lutwak, Yang and Zhang [10], the result is a direct application of Hölder’s inequality to \( \exp D_q(f\|g) \). Although [10] only strictly speaking considers the 1-dimensional case, the general case is precisely the same.

As for the Shannon maximisers, we use this Gibbs inequality Lemma 2.3 to show that the densities of Definition 2.1 really do maximise the Rényi entropy. (Note that this is an alternative proof to that given by Costa, Hero and Vignat [4], who introduced a non-symmetric directed divergence measure

\[
D_q(f\|g) = \text{sign}(q-1) \int_{\Omega_{q,C}} \frac{f^q(x)}{q} + \frac{q-1}{q} g^q(x) - f(x)g^{q-1}(x) dx.
\]

The approach of [4] is similar to that used by Cover and Thomas [5, p.234] in the Gaussian case. The general theory of directed divergence measures is discussed by Csiszar [6] and by Ali and Silvey [1]).
Proposition 2.4  Given any $q > n/(n + 2)$, and positive definite symmetric matrix $C$, among all probability densities $f$ with mean $0$ and $\int_{\Omega_q} f(x)xx^T dx = C$, the Rényi entropy is uniquely maximised by $g_{q,C}$, that is

$$H_q(f) \leq H_q(g_{q,C}),$$

with equality if and only if $f = g_{q,C}$ almost everywhere.

Proof  Since $f$ and $g_{q,C}$ have the same covariance matrix,

$$\int_{\Omega_q} (x^T C^{-1} x) f(x) dx = \int_{\Omega_q} (x^T C^{-1} x) g_{q,C}(x) dx.$$

This means that for $q \neq 1$

$$\int_{\Omega_q} g_{q,C}^{q-1}(x) f(x) dx = \int_{\Omega_q} A_q^{q-1} (1 - (q - 1)\beta x^T C^{-1} x) f(x) dx$$

$$= \int_{\Omega_q} A_q^{q-1} (1 - (q - 1)\beta x^T C^{-1} x) g_{q,C}(x) dx$$

$$= \int_{\Omega_q} g_{q,C}^q(x) dx. \quad (3)$$

For $q = 1$, the equivalent of the orthogonality property Equation (3) is the well-known fact that

$$\int f(x) \log g_{1,C}(x)d\mathbf{x} = \int f_1(x) \log g_{1,C}(x)d\mathbf{x}.$$

Using Equation (3) we simply evaluate

$$D_q(f\|g_{q,C}) = \frac{1}{1 - q} \log \left( \int g_{q,C}^{q-1}(x)f(x)d\mathbf{x} \right) + \frac{1 - q}{q} H_q(g_{q,C}) - \frac{1}{q} H_q(f)$$

$$= \frac{1}{q} \left( H_q(g_{q,C}) - H_q(f) \right),$$

and so the result follows by Lemma 2.3. \qed

Throughout this paper, we write $\chi_m$ for a random variable with density

$$f_m(x) = \frac{2^{1-m/2}}{\Gamma(m/2)} x^{m-1} \exp \left( -\frac{x^2}{2} \right), \text{ for } x > 0. \quad (4)$$
(Strictly speaking, this is only a $\chi$ random variable when the parameter $m$ is an integer, but it is simpler to adopt the convention of allowing non-integer $m$ than to refer to the square root of a $\Gamma(m)$ random variable). The Rényi maximisers have the following stochastic representations, and are linked by a duality relation.

**Proposition 2.5** Writing $R_{q,C}$ for a $n$-dimensional $q$-Rényi maximiser with mean 0 and covariance $C$, and writing $Z_C$ for a $N(0, C)$:

1. **Student-$r$.** For any $q > 1$, writing $m = n + 2q/(q - 1)$

   $$R_{q,C}U \sim Z_{mC},$$

   where $U \sim \chi_m$ (independent of $R_{q,C}$).

2. **Student-$t$.** For any $n/(n+2) < q < 1$, writing $m = 2/(1-q) - n > 2$

   $$R_{q,C} \sim Z_{(m-2)C}/U,$$

   where $U \sim \chi_m$ (independent of $Z$).

3. **Duality.** Given matrix $D$, define the map

   $$\Theta_D(x) = \frac{x}{\sqrt{x^T D^{-1} x + 1}},$$

   For $q < 1$, writing $m = 2/(1-q) - n$, if $R_{q,C}$ is a Rényi maximiser, then $\Theta_{C(m-2)}(R_{q,C}) \sim R_{p,C^*}$, where $1/(p - 1) = 1/(1-q) - n/2 - 1$ (so $q < 1$ implies that $p > 1$) and $C^* = C((m - 2)/(m + n))$.

**Proof** See Section 5. \hfill $\Box$

Stochastic representations (5) and (6) can be used to compute the covariance and entropy of $R_{q,C}$. For example, for $q < 1$, since $U \sim \chi_m$, the $E \frac{1}{U^2} = \frac{1}{m - 2}$, so that $\text{Cov} (R_{q,C}) = E Z_{(m-2)C} Z_{(m-2)C}^T E \frac{1}{U^2} = (m - 2)C \frac{1}{m - 2}$, as claimed.
Similarly for $q < 1$, the Shannon entropy $H_1(R_q, C)$ is given by (writing $m = 2/(1 - q) - n$)

$$-\mathbb{E} \log g_{q,C}(R_q, C) = - \log A_q + \frac{m + n}{2} \mathbb{E} \log \left( 1 + \frac{Z_{(m-2)C}^T C^{-1} Z_{(m-2)C}}{(m-2)U^2} \right)$$

$$= - \log A_q + \frac{m + n}{2} \mathbb{E} \log \left( 1 + \frac{N^T N}{U^2} \right)$$

$$= - \log A_q + \frac{m + n}{2} \mathbb{E} \left( \log \frac{2}{\chi_m^2 + \log \chi_m^2} \right)$$

where $N \sim N(0, I)$, and since $\mathbb{E} \log \chi_m^2 = \Psi \left( \frac{m}{2} \right)$, we obtain

$$H_1(R_q, C) = - \log A_q + \frac{1}{1 - q} \left( \Psi \left( \frac{1}{1 - q} \right) - \Psi \left( \frac{1}{1 - q} - \frac{n}{2} \right) \right). \quad (7)$$

Indeed, the theory of such stochastic representations can be generalized to multivariate maximizers with different powers. That is, given a sequence $(p_1, \ldots, p_n)$, the solution to the problem

$$\max H_q(X) \text{ such that } \mathbb{E} X_i^{p_i} = K_i, \text{ is a random vector } X \text{ with density given by}$$

$$f(x) \propto \left( 1 \pm \sum a_i x_i^{p_i} \right)^{\frac{1}{p_i - 1}},$$

where it can be shown that the $a_i$ all have the same size, and moreover, if $X$ is such a maximizer with $q > 1$, then $UX$ is Gaussian when $U$ has a $\chi$ distribution. This can be proved as in Section 5.

### 3 $\star$-convolution and relative entropy

For the sake of simplicity, we write $D(X \| Y)$ for the relative entropy between the two densities $f_X$ and $f_Y$ of random variables $X$ and $Y$. We define a new distance measure:

**Definition 3.1** Given a $n$-dimensional random vector $T$ with mean 0 and covariance $C$, we define its distance from a $n$-dimensional $q$-Rényi maximiser $R_q, C$ (for $q > 1$) to be

$$d(T|R_q, C) = D(TU \| Z),$$
where $U$ is a $\chi_m$ random variable (with $m = n + 2q/(q-1)$ degrees of freedom) independent of $T$, and $Z \sim N(0, mC)$.

Note that $d$ inherits positive definiteness from $D$ – that is $d(T|R_q,C) \geq 0$, with equality if and only if $T \sim R_q,C$. Note further that Equation (12) below implies that

$$d(T|R_q,C) = D(TU||R_q,CU) \leq D(T||R_q,C).$$

Motivated by Proposition 2.5, we make the following definition:

**Definition 3.2** For fixed $q > 1$, given two $n$-dimensional random vectors $S$, $T$, with covariance matrices $C(S)$ and $C(T)$, define the $\star$-convolution of $S$ and $T$ to be the $n$-dimensional random vector

$$S \star T = \Theta_{mC} \left( \frac{U(S)S + U(T)T}{V} \right) = \frac{(U(S)S + U(T)T)}{\sqrt{(U(S)S + U(T)T)^T(mC)^{-1}(U(S)S + U(T)T) + V^2}},$$

where $C = C(S) + C(T)$, and $U(S), U(T), V$ are independent $\chi$ random variables, where $U(S)$ and $U(T)$ have $m = n + 2q/(q-1)$ degrees of freedom, and $V$ has $2q/(q-1)$ degrees of freedom.

Again, notice that as $q \to 1$, $U(\cdot)/(2q/(q-1)) \to 1$ and $V/(2q/(q-1)) \to 1$ by the Law of Large Numbers, so $S \star T \overset{d}{\to} S + T$.

**Lemma 3.3** For $q > 1$, if $S$, $T$ are $q$-Rényi with covariances $C_S$, $C_T$ then $S \star T$ is also $q$-Rényi, with covariance $m/(m+n)(C_S + C_T)$.

**Proof** By Proposition 2.5.1, $W = \sqrt{(m-2)/m(U(S)S + U(T)T)}$ is $N(0, (m-2)C)$, where $C = C_S + C_T$. Then (by Proposition 2.5.2) $W/V$ is $q_1$-Rényi, with covariance $C$ where $1/(1-q_1) = 1 + 1/(q-1) + n/2$. Finally (by Proposition 2.5.3), $\Theta_{(m-2)C}(W/V)$ is $q_2$-Rényi, where $1/(q_2-1) = 1/(1-q_1) - n/2 - 1 = 1/(q-1)$, so in fact is $q$-Rényi with covariance $C(m-2)/(m+n)$. Now, $S \star T = \sqrt{m/(m-2)}\Theta_{(m-2)C}(W/V)$, so the result follows. \[\square\]

Now the classical Entropy Power Inequality implies a similar result for the $\star$-convolution.
Theorem 3.4 Given $q > 1$, for independent $n$-dimensional random vectors $S, T$ with mean 0 and covariances $C_S, C_T$,

$$|C_S + C_T|^{1/n} \exp \left(-2d(S * T \mid R_{q,C^*})/n \right) \geq |C_S|^{1/n} \exp \left(-2d(S \mid R_{q,C_S})/n \right) + |C_T|^{1/n} \exp \left(-2d(T \mid R_{q,C_T})/n \right),$$

with equality if and only if $S$ and $T$ are $q$-Rényi with proportional covariance matrices. Here $C^* = (C_S + C_T)m/ (m + n)$, with $m = n + 2q/(q - 1)$.

Proof We use the classical Entropy Power Inequality, which was first stated by Shannon as Theorem 15 of [12], with a ‘proof’ sketched in Appendix 6. More rigorous proofs appeared in Blachman [2] and later in Dembo, Cover and Thomas [7]. The result gives that for independent $n$-dimensional random vectors $X$ and $Y$,

$$\exp(2H(X + Y)/n) \geq \exp(2H(X)/n) + \exp(2H(Y)/n), \quad (8)$$

with equality if and only if $X$ and $Y$ are Gaussian with proportional covariance matrices.

Now, writing $C_X$ for the covariance matrix of $X$, we know that $D(X \mid Z_X) = (n \log(2\pi e) + \log |C_X|)/2 - H(X)$, so that the Entropy Power Inequality (8) is equivalent to

$$|C_X + C_Y|^{1/n} \exp \left(-2D(X + Y \mid Z_{C_X+C_Y})/n \right) \geq |C_X|^{1/n} \exp \left(-2D(X \mid Z_{C_X})/n \right) + |C_Y|^{1/n} \exp \left(-2D(Y \mid Z_{C_Y})/n \right). \quad (9)$$

By Proposition 5.3 we know that for $U^{(S)}, U^{(T)}, V, W$ all independent and $\chi$-distributed, where $U^{(S)}, U^{(T)}, W$ have $m = n + 2q/(q - 1)$ degrees of freedom, and $V$ has $2q/(q - 1)$ degrees of freedom:

$$d(S * T \mid R_{q,C^*}) = D((S * T)W \mid Z_{mC^*})$$

$$= D \left( \frac{(U^{(S)}S + U^{(T)}T)}{\sqrt{(U^{(S)}S + U^{(T)}T)^T C^{-1}(U^{(S)}S + U^{(T)}T) + V^2}} \left| Z_{mC^*} \right) \right) \leq D \left( U^{(S)}S + U^{(T)}T \mid Z_{mC_S+mC_T} \right). \quad (10)$$
We can combine Equations (9) and (10) to obtain that
\[
|mC_S + mC_T|^{1/n} \exp(-2d(S \ast T|R_q, C_\tau)/n) \\
\geq |mC_S + mC_T|^{1/n} \exp(-2D(U^{(S)}S + U^{(T)}T\|Z_{mC_S + mC_T})/n) \\
\geq |mC_S|^{1/n} \exp(-2D(U^{(S)}S\|Z_{mC_S})/n) \\
+ |mC_T|^{1/n} \exp(-2D(U^{(T)}T\|Z_{mC_T})/n) \\
= |mC_S|^{1/n} \exp(-2d(S|R_q, C_S)/n) + |mC_T|^{1/n} \exp(-2d(T|R_q, C_T)/n),
\]
and the result follows. Equality holds in Equation (10) if \(U^{(S)}S + U^{(T)}T\) is Gaussian. This, along with proportionality of covariance matrices, is also the condition for equality in Equation (9). \(\square\)

4 \(q\)-heat equation and \(q\)-Fisher information

We compute the exact constants in a result of Compte and Jou [3].

**Theorem 4.1** For a fixed \(\mu\), write \(f_\tau\) for the density of a \(R_{q,\tau\mu}C\) random variable. If \(\mu = 2/(2 + n(q - 1)/2)\) then \(f_\tau\) satisfies a heat equation of the form
\[
D_q \frac{\partial}{\partial \tau} f_\tau(x) = \sum_{k,l} C_{kl} \frac{\partial^2}{\partial x_k \partial x_l} f_\tau^q(x)
\]
with
\[
D_q = A_q^{-1} \frac{2q(2 + n(q - 1))}{2q + n(q - 1)}.
\]

**Proof** By Equation (2), we know that for a general choice of \(\mu\):
\[
f_\tau(x) = \frac{A_q}{\tau^{n\mu/2}} \left(1 - \frac{(q - 1)\beta x^T C^{-1} x}{\tau^\mu}\right)^{1/2}, \text{ where } \beta = \frac{1}{2q - n(1 - q)}.
\]
First note that
\[
\frac{\partial}{\partial \tau} f_\tau(x) = f_\tau(x) \left(-\frac{n\mu}{2\tau} + \frac{\beta \mu x^T C^{-1} x}{\tau^{\mu+1}} \left(1 - \frac{(q - 1)\beta x^T C^{-1} x}{\tau^\mu}\right)^{-1}\right). \ (11)
\]
Further, for any $k$, writing $\mathbf{A} = \mathbf{C}^{-1}$:

$$\frac{\partial}{\partial x_k} f^q_\tau (\mathbf{x}) = \frac{A_q^q}{\tau^{nq\mu/2}} \left( 1 - \frac{(q-1)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}{\tau^\mu} \right)^{\frac{1}{\tau^\mu}} \left( -\frac{2q\beta (\mathbf{A}x)_k}{\tau^\mu} \right).$$

Hence, for any $k, l$:

$$\frac{\partial^2}{\partial x_k \partial x_l} f^q_\tau (\mathbf{x}) = \frac{A_q^q}{\tau^{nq\mu/2}} \left( 1 - \frac{(q-1)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}{\tau^\mu} \right)^{\frac{1}{\tau^\mu}} \left( -\frac{2q\beta A_{kl}}{\tau^\mu} \right)
+ \frac{A_q^q}{\tau^{nq\mu/2}} \left( 1 - \frac{(q-1)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}{\tau^\mu} \right)^{\frac{1}{\tau^\mu} - 1} \left( \frac{4q^2 \beta^2}{\tau^{2\mu}} (\mathbf{A}x)_k (\mathbf{A}x)_l \right)
= \frac{A_q^q}{\tau^{n(q-1)\mu/2}} f_\tau (\mathbf{x}) \left( -\frac{2q\beta A_{kl}}{\tau^\mu} + \frac{4q\beta^2 (\mathbf{A}x)_k (\mathbf{A}x)_l}{\tau^{2\mu}} \left( 1 - \frac{(q-1)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}{\tau^\mu} \right)^{-1} \right)$$

Overall, we deduce that

$$\sum_{k,l} C_{kl} \frac{\partial^2}{\partial x_k \partial x_l} f^q_\tau (\mathbf{x}) = \frac{A_q^q}{\tau^{n(q-1)\mu/2}} f_\tau (\mathbf{x}) \left( -\frac{2q\beta n}{\tau^\mu} + \frac{4q\beta^2 \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}{\tau^{2\mu}} \left( 1 - \frac{(q-1)\beta \mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}}{\tau^\mu} \right)^{-1} \right)$$

so that equating this with Equation (11) we obtain:

$$D_q = \frac{A_q^{q-1}}{\tau^{n(q-1)\mu/2 + \mu - 1}} \frac{4q\beta}{\mu}.$$ 

Now, we want this to not be a function of $\tau$, so take $\mu = 2/(2 + n(q - 1))$, and substitute for $\beta$ to obtain

$$D_q = A_q^{q-1} \frac{2q(2 + n(q - 1))}{2q + n(q - 1)},$$

as claimed.

Note that value of the exponent $\mu$ coincides with the one given by Compte and Jou [3]. Further, as $\lim_{q \to 1} A_q^{q-1} = 1$, so that $\lim_{q \to 1} D_q = 2$, as we would
expect from the de Bruijn identity given in Lemma 2.2 of Johnson and Suhov [9]. We can evaluate the derivative of the Rényi entropy:

\[
\frac{\partial}{\partial \tau} H_q(f_\tau) = \frac{1}{1-q} \frac{(q-1) \int f_\tau(x)^{q-1} \frac{\partial}{\partial \tau} f_\tau(x) dx}{\int f_\tau(x)^q dx}
\]

\[
= -\frac{D_q^{-1}}{\int f_\tau(x)^q dx} \sum_{k,l} C_{kl} \int f_\tau(x)^{q-1} \frac{\partial^2}{\partial x_k \partial x_l} f_\tau^q(x) dx
\]

\[
= \frac{D_q^{-1}}{\int f_\tau(x)^q dx} \sum_{k,l} C_{kl} \int \frac{\partial}{\partial x_l} f_\tau(x)^{q-1} \frac{\partial}{\partial x_k} f_\tau^q(x) dx
\]

\[
= \frac{D_q^{-1} q(q-1)}{\int f_\tau(x)^q dx} \sum_{k,l} C_{kl} \int f_\tau(x)^{2q-3} \frac{\partial}{\partial x_l} f_\tau(x) \frac{\partial}{\partial x_k} f_\tau(x) dx
\]

\[
= D_q^{-1} q(q-1) \text{tr}(C J_q(f_\tau)),
\]

where we make the following definitions:

**Definition 4.2** Given probability density \( p \), define the \( q \)-score function

\[
\rho_q(x) = \nabla p(x)/p(x)^{2-q},
\]

and the \( q \)-Fisher information matrix to be

\[
J_q(p) = \frac{\int p(x) \rho_q^T(x) \rho_q(x) dx}{\int p(x)^q dx}.
\]

Note that the denominator is the case \( p = 2, \lambda = q \) of the \((p, \lambda)\) Fisher information introduced in Equation (7) of [10]. We establish a multi-dimensional Cramér-Rao inequality:

**Proposition 4.3** For the Fisher information \( J_q \) defined above, if random variable \( p \) has covariance \( C \) then

\[
J_q(p) - \frac{\int p(x)^q dx}{q^2} C^{-1}
\]

is positive definite, with equality if and only if \( p = g_q C \) everywhere.
Proof The key is a Stein-like identity, as usual found using integration by parts, since
\[ \int p(x)(\rho_q(x))_I(Ax)_k dx = \int \frac{\partial}{\partial x_i} p(x) p^{q-1}(x) (Ax)_k dx \]
\[ = \frac{1}{q} \int \frac{\partial}{\partial x_i} (p^q(x)(Ax)_k) dx \]
\[ = -\frac{1}{q} \int p^q(x) A_{kl} dx. \]

This means that for any \( c \), the positive definite matrix
\[ \int p(x)(\rho_q(x) + cAx)^T(\rho_q(x) + cAx) \]
\[ = \int p(x)\rho_q^T(x)\rho_q(x)dx + 2\frac{c}{q} \int p^q(x)dx + c^2 A. \]

So we choose \( c = \left( \int p^q(x)dx \right) / q \), and the result follows. Note that equality holds if and only if \( p = g_q c \) everywhere, since the Rényi maximiser has score function \( \rho(x) = A_q^{-1}(-2\beta)Ax \), and \( \int g_q^q c(x)dx / q = \int g_q c A_q^{-1}(1 - \beta(q - 1)n)q = A_q^{-1}(2\beta). \) \( \square \)

Now, we can give the extensivity property for Fisher information defined in this way:

**Lemma 4.4** For a compound system of independent random vectors \( X \) and \( Y \), for \( q > 1/2 \) the \( q \)-Fisher information satisfies:
\[ J_q(X, Y) = \begin{pmatrix} \alpha_q(Y)J_q(X) & 0 \\ 0 & \alpha_q(X)J_q(Y) \end{pmatrix}, \]
where constant \( \alpha_q(X) = \left( \int p^q_X(x)^{2q-1}dx \right) / \left( \int p^q_X(x)^qdx \right) \) and \( \alpha_q(Y) \) similarly.

Proof We write \( p_{X,Y}(x,y) = p_X(x)p_Y(y) \), so that (omitting the arguments for clarity), we can express
\[ \nabla p_{X,Y} = (p_Y \nabla p_X, p_X \nabla p_Y). \]
Then
\[
\int \int p_{X,Y}^{2q-3} \nabla^T p_{X,Y} \nabla p_{X,Y} = \left( \int \int p_Y^{2q-1} p_X^{2q-3} \nabla^T p_X \nabla p_X \right) \left( \int \int p_X^{2q-1} p_Y^{2q-3} \nabla^T p_Y \nabla p_Y \right)
\]
\[
= \left( \int p_Y^{2q-1} \int p_X^q J_q(X) \right) \left( \int p_X^{2q-1} \int p_Y^q J_q(Y) \right),
\]
since for \( q > 1/2 \), the off-diagonal term
\[
\left( \int p_X^{2q-2} \nabla p_X \right) \left( \int p_Y^{2q-2} \nabla p_Y \right)
\]
\[
= \frac{1}{(2q-1)^2} \left( \int \nabla p_X^{2q-1} \right) \left( \int \nabla p_Y^{2q-1} \right) = 0,
\]
since this is a perfect derivative, and since \( p_X(x) \to 0 \) as \( x \to \infty \). The result follows since
\[
\int \int p_{X,Y}^q = \left( \int p_X^q \right) \left( \int p_Y^q \right).
\]

5 Proofs

5.1 Stochastic Representation

Proof of Proposition 2.5
1. By Equation (4), since we take \( \beta(q-1) = 1/m \) in Equation (2), the density of \( R_{q,C}U \) can be expressed as

\[
g(y) = \frac{2^{1-m/2}}{\Gamma(m/2)} \int_0^\infty \frac{1}{x^m} \left( 1 - \frac{y^T C^{-1} y}{mx^2} \right)^{\frac{1}{2}q-1} x^{m-1} \exp \left( -\frac{x^2}{2} \right) dx
\]
\[
= \frac{2^{1-m/2}}{\Gamma(m/2)} A_q \exp \left( -\frac{y^T C^{-1} y}{2m} \right) K.
\]
Here since \( m-n-2 = 2/(q-1) \), taking \( u^2 = x^2 - y^T C^{-1} y/m \), so \( udu = xdx \):

\[
K = \int_0^\infty \left( 1 - \frac{y^T C^{-1} y}{mx^2} \right)^{\frac{q}{2m}} x^{m-n-2} \exp \left( -\frac{x^2}{2} + \frac{y^T C^{-1} y}{2m} \right) xdx
\]

\[
= \int_0^\infty u^{2q/m} \exp \left( -\frac{u^2}{2} \right) udu = 2^{\frac{1}{2-q}} \Gamma \left( \frac{q}{q-1} \right),
\]

and the result follows, since the constant

\[
\frac{2^{1-m/2}}{\Gamma(m/2)} A_q K = \frac{1}{(2\pi m)^{n/2} |C|^{\frac{1}{2}}},
\]

since \( 1 - m/2 + 1/(q-1) = -n/2 \) and \( \beta(q-1) = 1/m \).

2. In the same way, the density of \( Z_{(m-2)C/U} \) can be expressed as

\[
\frac{2^{1-q/2}}{\Gamma(m/2)} \int_0^\infty \frac{x^n}{(2\pi(m-2))^{n/2} |C|} \exp \left( -\frac{x^2y^T C^{-1} y}{2(m-2)} \right) x^{m-1} \exp \left( -\frac{x^2}{2} \right) dx
\]

\[
= \frac{2^{1-q/2}}{\Gamma(m/2)} \sqrt{\frac{\beta(1-q)}{\pi^n |C|}} \int_0^\infty \exp \left( -(1+d)x^2 \right) x^{n+m-1} dx
\]

\[
= \frac{\Gamma((n+m)/2)}{\Gamma(m/2)} \sqrt{\frac{\beta(1-q)}{\pi^n |C|}} (1+d)^{-\frac{n+m}{2}}
\]

\[
= \frac{\Gamma(1/(1-q))}{\Gamma(1/(1-q) - n/2)} \sqrt{\frac{\beta(1-q)}{\pi^n |C|}} \left( 1 + \frac{y^T C^{-1} y}{m-2} \right)^{\frac{1}{2-q}},
\]

writing \( d = 1 + (y^T C^{-1} y)/(m-2) \), and using the facts that \( (m+n)/2 = 1/(1-q) \) and \( 1/(m-2) = \beta(1-q) \), the result follows.

3. For this choice of parameters, \( X = R_{q,C} \) has density \( A_q (1+x^T D^{-1} x)^{1/(q-1)} \).

If \( Y = \Theta_D(X) \), we can calculate the Jacobian \( |\partial X/\partial Y| = (1-Y^T D^{-1} Y)^{-1-n/2} \).

Then, the standard change-of-variables relation gives that, since \( 1-Y^T D^{-1} Y = (1+X^T D^{-1} X)^{-1} \), we know that \( Y \) has density

\[
g_Y(y) = (1-y^T D^{-1} y)^{-1-\frac{q}{2}} g_X(\Theta_D^{-1}(y)).
\]

Thus, in particular, taking \( X \sim R_{q,C} \) and \( D = C(m-2) \), we know that

\[
g_Y(y) = (1-y^T D^{-1} y)^{-1-\frac{q}{2}} A_q (1-y^T D^{-1} y)^{-\frac{1}{2-q}}
\]

\[
= A_q (1-y^T D^{-1} y)^{\frac{1}{2-q}}.
\]
Since \( p > 1 \), we know that \( Y \) has covariance \( D \beta_p(p - 1) = D/(2p/(p - 1) + n)^{-1} = C(m - 2)/(m + n) \).

Further \( A_q = \left( \beta_q(1 - q) \right)^{n/2} \Gamma \left( \frac{1}{1-q} \right) / \left( \Gamma \left( \frac{1}{1-q} - \frac{n}{2} \right) \pi^{n/2} |C|^{1/2} \right) \)

\[
= \left( \beta_p(p-1) \right)^{n/2} \Gamma \left( \frac{1}{p-1} + \frac{n}{2} + 1 \right) / \left( \Gamma \left( \frac{1}{p-1} + 1 \right) \pi^{n/2} (m-2)/(m+n) |C|^{1/2} \right) = A_p, \text{ as required.}\]

### 5.2 Projection results

To prove the Entropy Power Inequality, Theorem 3.4, we prove a technical result, Proposition 5.3. This in turn relies on two simple technical results, Lemma 5.1 and Lemma 5.2 which are proved first. Firstly as a consequence of the chain rule for relative entropy (see for example Theorem 2.5.3 of Cover and Thomas [5]):

**Lemma 5.1** For pairs of random variables \((X,Y)\) and \((U,V)\),

\[
D((X,Y)\| (U,V)) \geq D(X\| U).
\]

Equality holds if and only if for each \(x\), the random variables \(Y|X = x\) and \(V|U = x\) have the same distribution. In particular if \((X,Y)\) and \((U,V)\) are independent pairs, equality holds if and only if \(Y\) and \(V\) have the same distribution.

Secondly, we state a projection identity.

**Lemma 5.2** For random variables \(X\) and \(Y\), and for any invertible function \(\Phi\):

\[
D(\Phi(X)\|\Phi(Y)) = D(X\|Y).
\]

**Proof** We can express the density of

\[
f_{\Phi(X)}(t) = |J(t)| f_X(\Phi^{-1}(t)), \quad f_{\Phi(Y)}(t) = |J(t)| f_Y(\Phi^{-1}(t)),
\]

where \(J\) is the Jacobian of \(\Phi^{-1}\). Then

\[
D(\Phi(X)\|\Phi(Y)) = \int |J(t)| f_X(\Phi^{-1}(t)) \log \left( \frac{|J(t)| f_X(\Phi^{-1}(t))}{|J(t)| f_Y(\Phi^{-1}(t))} \right) dt
\]

\[
= \int f_X(s) \log \left( \frac{f_X(s)}{f_Y(s)} \right) ds = D(X\|Y),
\]

as required. \(\square\)
Proposition 5.3 For a $n$-dimensional random vector $M$, take $N \sim \chi^2_{q/(q-1)}$ and $U \sim \chi^2_{q/(q-1)+n}$, where $(M, N, U)$ are independent:

$$D\left(\frac{M}{\sqrt{M^T C^{-1} M + N^2}} U \bigg| \frac{Z}{\sqrt{Z^T C^{-1} Z + Y^2}}\right) \leq D(M \| Z),$$

where $C^* = Cm/(m+n)$, and equality holds if $M$ is $N(0, C)$.

**Proof** By combining Lemmas 5.1 and 5.2, if $Y, V$ have the same distribution and $(X, Y)$ and $(U, V)$ each form independent pairs then

$$D(XY \| UV) \leq D((XY, Y) \| (UV, V)) = D((X, Y) \| (U, V)) = D(X \| U).$$

(12)

We define $V \sim \chi^2_{q/(q-1)}$ and $Y \sim \chi^2_{q/(q-1)+n}$, both independent of $Z_C$, so that the LHS becomes:

$$D\left(\frac{M}{\sqrt{M^T C^{-1} M + N^2}} U \bigg| \frac{Z}{\sqrt{Z^T C^{-1} Z + Y^2}}\right) \leq D\left(\frac{M}{\sqrt{M^T C^{-1} M + N^2}} \bigg| \frac{Z}{\sqrt{Z^T C^{-1} Z + Y^2}}\right)$$

(13)

$$= D\left(\Theta_C(M/N) \| \Theta_C(Z_C/Y)\right)$$

(14)

$$\leq D(M \| Z_C),$$

(15)

and the result follows. Here Equation (13) follows by Equation (12), Equation (14) follows by Lemma 5.2 and Equation (15) again follows by Equation (12). □
References


**Addresses:**

Oliver Johnson, Statistical Laboratory, DPMMS, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK. Fax: +44 1223 337956. Email: otj1000@cam.ac.uk.

Christophe Vignat L.I.S., 961 rue de la Houille Blanche, 38402 St. Martin d’Hères cedex, France. Email: vignat@univ-mlv.fr.