ADVANCED FINANCIAL MODELS

Attempt **FOUR** questions.

There are **six** questions in total.

*Marks for each question are indicated on the paper in square brackets.*

*Each question is worth a total of 20 marks.*

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**STATIONERY REQUIREMENTS**

- Cover sheet
- Treasury Tag
- Script paper

**SPECIAL REQUIREMENTS**

- None

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You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
1 Write an essay on optimal hedging in the least-squares sense in a one-period financial model. Your essay should cover the notions of attainable claims, dominated and equivalent martingale measures, the minimal martingale measure and a proof of the fact that the model is complete if and only if there is a unique dominated martingale measure. [20]

2 Consider the standard binomial model operating over the times $0, 1, \ldots, n$ ($n \geq 2$) where the stock price at time $r$ is denoted by $S_r$. Let $g_r(S_r)$ represent the price at time $r$ of a claim which pays $f(S_n)$ at time $n$. When $f$ is convex show that $g_r$ is convex on the possible values that $S_r$ can take on (viz. $S_r = S_0 u^i d^{r-i}$, $i = 0, 1, \ldots, r$). [6]

Show that when $f$ is convex then the amount of stock held in the hedging portfolio increases between the times $r$ and $r + 1$ ($< n$) if the stock price increases between $r$ and $r + 1$. [8]

Now consider an investor who has initial wealth $w_0 > 0$, at time 0, and utility function $v(x) = \gamma x^{1/\gamma}$, for $x > 0$, where $\gamma > 1$; determine the claim that he would purchase in order to maximize the expected utility of his final wealth. [6]

3 Let $T_{a,b}$ denote the first hitting time of the line $a + bs$ by a standard Brownian motion, where $a > 0$ and $-\infty < b < \infty$ and let $T_n = T_{a,0}$ represent the first hitting time of the level $a$.

For $\theta > 0$, using the fact that $\mathbb{E}(e^{-\theta T_n}) = e^{-a\sqrt{\theta}}$ or otherwise, derive an expression for $\mathbb{E}(e^{-\theta T_{a,b}})$ for each $b$, $-\infty < b < \infty$. [8]

Hence, or otherwise, show that, for $t > 0$,

$$
\mathbb{P}(T_{a,b} \leq t) = e^{-2ab} \Phi \left( \frac{bt - a}{\sqrt{t}} \right) + 1 - \Phi \left( \frac{a + bt}{\sqrt{t}} \right),
$$

where $\Phi$ is the standard normal distribution function. [6]

Use this result, in the context of the Black–Scholes model, to derive the price at time 0 of a barrier digital put which pays 1 at time $t_0$ if and only if the stock price stays below a predetermined barrier $c > S_0$ between times 0 and $t_0$, where $S_0$ is the initial price of the stock. [6]
4 Suppose that in the Black–Scholes model, the stock price at time $t$ is $S_t$, the fixed interest rate is $\rho$ and the volatility is $\sigma$. Let $p(S_t, t)$ be the price at time $t$ of a claim paying $C = f(S_{t_0})$ at time $t_0$; explain carefully why the function $p = p(x, t)$ satisfies the Black–Scholes equation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} + \rho x \frac{\partial p}{\partial x} - \frac{\partial p}{\partial t} - \rho p = 0. \quad [12]$$

Now suppose that in addition to paying $C$ at time $t_0$ the claim pays a dividend at rate $R_d = k(S_t, t)$ at time $t$. Explain how the Black–Scholes equation for the price of the claim $p(S_t, t)$ should be modified in this case. Justify your answer carefully. \[8\]

5 For the Black–Scholes model, give a description of the pricing of a terminal-value claim paying the amount $f(S_{t_0})$ at time $t_0$, where $\{S_t, t \geq 0\}$ is the stock price process. You may assume that $f$ is a twice-differentiable function and your account should include a verification that the price satisfies the Black–Scholes equation as well as an analysis of its dependence on the various parameters of the model. \[16\]

In particular, show that if $f$ is convex and the replicating portfolio is short in bonds (that is, it holds a negative amount) then the price is a decreasing function of time. \[4\]

6 Write an essay on modelling interest rates with Gaussian random fields. You need not include detailed proofs of results but you should outline how they are obtained. \[20\]

END OF PAPER