Mathematical Tripos: Part IB **DJS/Lent 2017**

Statistics: Example Sheet 2 (of 3)

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1. Let X have density function $f(x;\theta) = \frac{\theta}{(x+\theta)^2}$, $x > 0$, where $\theta \in (0, \infty)$ is an unknown parameter. Find the likelihood ratio test of size 0.05 of $H_0 : \theta = 1$ against $H_1 : \theta = 2$, and show that the probability of Type II error is 19/21.

2. Let $X_1, X_2, \ldots, X_n$ be iid random variables, each with a Poisson distribution with parameter θ (and therefore with mean θ and variance θ). Find the form of the likelihood ratio test of $H_0 : \theta = 1$ against $H_1 : \theta = 1.21$. By using the Central Limit Theorem to approximate the distribution of $\sum_i X_i$, show that the smallest value of n required to make $\alpha = 0.05$ and $\beta \leq 0.1$ (where α and β are the Type I and Type II error probabilities) is somewhere near 212.

3. Let f_0 and f_1 be probability mass functions for $\mathbf{X} = (X_1 \ldots, X_n)$ on a countable set $\mathcal{X}^n$. State and prove a version of the Neyman–Pearson lemma for a size α test of $H_0 : f = f_0$ against $H_1 : f = f_1$, assuming that α is such that there exists a likelihood ratio test of exact size α.

4. Let $X \sim \text{Bin}(2, \theta)$ and consider testing $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta = \frac{3}{4}$. Find the possible values of α for which there exists a likelihood ratio test with size exactly α.

5. Let $X_1, \ldots, X_n$ be iid random variables each with a $N(\mu_0, \sigma^2)$ distribution, where μ_0 is known and σ^2 is unknown. Find the best (most powerful) test of size at most α for testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$ for known σ_0^2 and σ_1^2 $(> \sigma_0^2)$. Show that this test is a size α uniformly most powerful test for testing $H_0' : \sigma^2 \leq \sigma_0^2$ against $H_1' : \sigma^2 > \sigma_0^2$.

6. Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exponential}(\theta)$. Find the likelihood ratio test of size α of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ $(> \theta_0)$ and derive an expression for the power function. Is the test uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$? Is it uniformly most powerful for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$?

7. Let $X_1, \ldots X_n, Y_1, \ldots, Y_n$ be independent, with $X_1, \ldots, X_n \sim \text{Exponential}(\theta_1)$ and $Y_1, \ldots, Y_n \sim \text{Exponential}(\theta_2)$. Recalling the forms of the relevant MLEs from Sheet 1, show that the likelihood ratio of $H_0 : \theta_1 = \theta_2$ and $H_1 : \theta_1 \neq \theta_2$ is a monotone function of $|t - 1/2|$, where t is the observed value of the statistic T given by

$$T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i}.$$

By writing down the distribution of T under H_0, express the likelihood ratio test of size α in terms of $|T - 1/2|$ and the percentage points of a beta distribution.
Hint: use Question 2 on Example Sheet 1.

8. A machine produces plastic articles (many of which are defective) in bunches of three articles at a time. Under the null hypothesis that each article has a constant (but unknown) probability θ of being defective, write down the probabilities $p_i(\theta)$ of a bunch having i defective articles, for $i = 0, 1, 2, 3$. In an trial run in which 512 bunches were produced, the numbers of bunches with i defective articles were 213 ($i = 0$), 228 ($i = 1$), 57 ($i = 2$) and 14 ($i = 3$). Carry out Pearson's chi-squared test at the 5% level of the null hypothesis, explaining carefully why the test statistic should be referred to the χ^2_2 distribution.

9. A random sample of 59 people from the planet Krypton yielded the results below.

		Eye-colour	
		1 (Blue)	2 (Brown)
Sex	1 (Male)	19	10
	2 (Female)	9	21

Carry out Pearson's chi-squared test at the 5% level of the null hypothesis that sex and eye-colour are independent factors on Krypton. Now carry out the corresponding test at the 5% level of the null hypothesis that each of the cell probabilities is equal to 1/4. Comment on your results.

10. The following data were published in a 1986 medical paper (doi:10.1136/bmj.292.6524.879) comparing two treatments for kidney stones (a very painful condition), where the 'Old' treatment is a standard surgical operation, and the 'New' is 'keyhole' surgery. Exactly 350 patients were chosen to receive each treatment.

	Success	Failure	Total
Old	273 (78%)	77	350
New	289 (83%)	61	350
Total	562 (80%)	138	700

Test the hypothesis that the old and new treatments are equally successful. On this basis, which treatment would you prefer?

Closer examination reveals that the treatments were not given at random, but patients were deliberately selected for the 'New' treatment. When the patients are split into those with 'Small' (< 2 cm) and those with 'Large' (≥ 2 cm) stones, the following results were found.

	Small stones		
	Success	Failure	Total
Old	81 (93%)	6	87
New	234 (87%)	36	270
Total	315 (88%)	42	357

	Large 77stones		
	Success	Failure	Total
Old	192 (73%)	71	263
New	55 (69%)	25	80
Total	247 (72%)	96	343

Test the hypothesis that the treatments are equally effective in small stones, and the hypothesis that the treatments are equally effective in large stones. How would you combine these test statistics into a single test of the hypothesis that the treatments are equally effective in Small stones, *and* equally effective in Large stones? Do you wish to revise your conclusion about the relative effectiveness of the two treatments?

How might you express these findings in words to a member of your family with minimal understanding of statistics?

$^{+}$11 In Question 3, does there exist a version of the Neyman–Pearson lemma when a likelihood ratio test of exact size α does not exist?