Lecture 8. Composite hypotheses
Composite hypotheses, types of error and power

- For composite hypotheses like $H: \theta \geq 0$, the error probabilities do not have a single value.

- Define the **power function** $W(\theta) = \mathbb{P}(X \in C | \theta) = \mathbb{P}(\text{reject } H_0 | \theta)$.

- We want $W(\theta)$ to be small on $H_0$ and large on $H_1$.

- Define the **size** of the test to be $\alpha = \sup_{\theta \in \Theta_0} W(\theta)$.

- For $\theta \in \Theta_1$, $1 - W(\theta) = \mathbb{P}(\text{Type II error} | \theta)$.

- Sometimes the Neyman–Pearson theory can be extended to one-sided alternatives.

- For example, in Example 7.3 we have shown that the most powerful size $\alpha$ test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu = \mu_1$ (where $\mu_1 > \mu_0$) is given by $C = \{x : \sqrt{n}(\bar{x} - \mu_0)/\sigma_0 > z\alpha \}$.

- This critical region depends on $\mu_0, n, \sigma_0, \alpha$, on the fact that $\mu_1 > \mu_0$, but not on the particular value of $\mu_1$. 
• Hence this $C$ defines the most powerful size $\alpha$ test of $H_0 : \mu = \mu_0$ against any $\mu_1$ that is larger than $\mu_0$.

• This test is then uniformly most powerful size $\alpha$ for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$.

**Definition 8.1**

A test specified by a critical region $C$ is **uniformly most powerful** (UMP) size $\alpha$ test for testing $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$ if

(i) $\sup_{\theta \in \Theta_0} W(\theta) = \alpha$;
(ii) for any other test $C^*$ with size $\leq \alpha$ and with power function $W^*$ we have $W(\theta) \geq W^*(\theta)$ for all $\theta \in \Theta_1$.

• UMP tests may not exist.

• However likelihood ratio tests are often UMP.
Example 8.2

Suppose $X_1, \ldots, X_n$ are iid $N(\mu_0, \sigma_0^2)$ where $\sigma_0$ is known, and we wish to test $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$.

- First consider testing $H'_0 : \mu = \mu_0$ against $H'_1 : \mu = \mu_1$ where $\mu_1 > \mu_0$ (as in Example 7.3).
- As in Example 7.3, the Neyman-Pearson test of size $\alpha$ of $H'_0$ against $H'_1$ has $C = \{x : \sqrt{n}(\bar{x} - \mu_0)/\sigma_0 > z_\alpha \}$.
- We will show that $C$ is in fact UMP for the composite hypotheses $H_0$ against $H_1$.
- For $\mu \in \mathbb{R}$, the power function is

$$W(\mu) = \mathbb{P}_\mu(\text{reject } H_0) = \mathbb{P}_\mu \left( \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma_0} > z_\alpha \right)$$

$$= \mathbb{P}_\mu \left( \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma_0} > z_\alpha + \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} \right)$$

$$= 1 - \Phi \left( z_\alpha + \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} \right).$$


\[
power = 1 - \text{pnorm}( \text{qnorm}(0.95) + \sqrt{n} \times (\mu_0 - x) / \sigma_0 )
\]
We know $W(\mu_0) = \alpha$. (just plug in)

$W(\mu)$ is an increasing function of $\mu$.

So $\sup_{\mu \leq \mu_0} W(\mu) = \alpha$, and (i) is satisfied.

For (ii), observe that for any $\mu > \mu_0$, the Neyman Pearson size $\alpha$ test of $H'_0$ vs $H'_1$ has critical region $C$ (the calculation in Example 7.3 depended only on the fact that $\mu > \mu_0$ and not on the particular value of $\mu_1$.)

Let $C^*$ and $W^*$ belong to any other test of $H_0$ vs $H_1$ of size $\leq \alpha$

Then $C^*$ can be regarded as a test of $H'_0 : \mu = \mu_0$ vs $H'_1$ of size $\leq \alpha$, and NP-Lemma says that $W^*(\mu_1) \leq W(\mu_1)$

This holds for all $\mu_1 > \mu_0$ and so (ii) is satisfied.

So $C$ is UMP size $\alpha$ for $H_0$ vs $H_1$. □
Generalised likelihood ratio tests

- We now consider likelihood ratio tests for more general situations.
- Define the **likelihood of a composite hypothesis** $H : \theta \in \Theta$ given data $x$ to be
  
  $$L_x(H) = \sup_{\theta \in \Theta} f(x|\theta).$$

- So far we have considered disjoint hypotheses $\Theta_0$, $\Theta_1$, but often we are not interested in any specific alternative, and it is easier to take $\Theta_1 = \Theta$ rather than $\Theta_1 = \Theta \setminus \Theta_0$.
- Then
  $$\Lambda_x(H_0; H_1) = \frac{L_x(H_1)}{L_x(H_0)} = \frac{\sup_{\theta \in \Theta_1} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)} (\geq 1),$$
  (1)
  
  with large values of $\Lambda_x$ indicating departure from $H_0$.
- Notice that if $\Lambda_x^* = \sup_{\theta \in \Theta \setminus \Theta_0} f(x|\theta) / \sup_{\theta \in \Theta_0} f(x|\theta)$, then
  $$\Lambda_x = \max\{1, \Lambda_x^*\}.$$
Example 8.3

Single sample: testing a given mean, known variance (z-test). Suppose that $X_1, \ldots, X_n$ are iid $N(\mu, \sigma_0^2)$, with $\sigma_0^2$ known, and we wish to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ ($\mu_0$ is a given constant).

- Here $\Theta_0 = \{\mu_0\}$ and $\Theta = \mathbb{R}$.
- For the denominator in (1) we have $\sup_{\Theta_0} f(x | \mu) = f(x | \mu_0)$.
- For the numerator, we have $\sup_{\Theta} f(x | \mu) = f(x | \hat{\mu})$, where $\hat{\mu}$ is the mle, so $\hat{\mu} = \bar{x}$ (check).
- Hence

$$\Lambda_x(H_0; H_1) = \frac{(2\pi \sigma_0^2)^{-n/2} \exp \left( -\frac{1}{2\sigma_0^2} \sum (x_i - \bar{x})^2 \right)}{(2\pi \sigma_0^2)^{-n/2} \exp \left( -\frac{1}{2\sigma_0^2} \sum (x_i - \mu_0)^2 \right)},$$

and we reject $H_0$ if $\Lambda_x$ is ‘large.’

- We find that

$$2 \log \Lambda_x(H_0; H_1) = \frac{1}{\sigma_0^2} \left[ \sum (x_i - \mu_0)^2 - \sum (x_i - \bar{x})^2 \right] = \frac{n}{\sigma_0^2} (\bar{x} - \mu_0)^2.$$  (check)

- Thus an equivalent test is to reject $H_0$ if $|\sqrt{n}(\bar{x} - \mu_0) / \sigma_0|$ is large.
8. Composite hypotheses

8.2. Generalised likelihood ratio tests

- Under $H_0$, $Z = \sqrt{n}(\bar{X} - \mu_0)/\sigma_0 \sim N(0, 1)$ so the size $\alpha$ generalised likelihood test rejects $H_0$ if $|\sqrt{n}(\bar{X} - \mu_0)/\sigma_0| > z_{\alpha/2}$.

- Since $n(\bar{X} - \mu_0)^2/\sigma_0^2 \sim \chi^2_1$ if $H_0$ is true, this is equivalent to rejecting $H_0$ if $n(\bar{X} - \mu_0)^2/\sigma_0^2 > \chi^2_1(\alpha)$ (check that $z_{\alpha/2}^2 = \chi^2_1(\alpha)$). □

Notes:

- This is a 'two-tailed' test - i.e. reject $H_0$ both for high and low values of $\bar{X}$.
- We reject $H_0$ if $|\sqrt{n}(\bar{X} - \mu_0)/\sigma_0| > z_{\alpha/2}$. A symmetric 100(1 $\alpha$)% confidence interval for $\mu$ is $\bar{X} \pm z_{\alpha/2} \sigma_0/\sqrt{n}$. Therefore we reject $H_0$ iff $\mu_0$ is not in this confidence interval (check).
- In later lectures the close connection between confidence intervals and hypothesis tests is explored further.
The 'generalised likelihood ratio test'

- The next theorem allows us to use likelihood ratio tests even when we cannot find the exact relevant null distribution.
- First consider the 'size' or 'dimension' of our hypotheses: suppose that $H_0$ imposes $p$ independent restrictions on $\Theta$, so for example, if and we have
  - $H_0 : \theta_{i_1} = a_1, \ldots, \theta_{i_p} = a_p \ (a_1, \ldots, a_p \text{ given numbers})$,
  - $H_0 : A\theta = b \ (A \ p \times k, \ b \ p \times 1 \text{ given})$,
  - $H_0 : \theta_i = f_i(\phi), \ i = 1, \ldots, k, \ \phi = (\phi_1, \ldots, \phi_{k-p})$.
- Then $\Theta$ has 'k free parameters' and $\Theta_0$ has 'k – p free parameters.'
- We write $|\Theta_0| = k - p$ and $|\Theta| = k$. 

Theorem 8.4

(not proved)
Suppose $\Theta_0 \subseteq \Theta_1$, $|\Theta_1| - |\Theta_0| = p$. Then under regularity conditions, as $n \to \infty$, with $\mathbf{X} = (X_1, \ldots, X_n)$, $X_i$’s iid, we have, if $H_0$ is true,

$$2 \log \Lambda_{\mathbf{X}}(H_0; H_1) \sim \chi^2_p.$$

If $H_0$ is not true, then $2 \log \Lambda$ tends to be larger. We reject $H_0$ if $2 \log \Lambda > c$ where $c = \chi^2_p(\alpha)$ for a test of approximately size $\alpha$.

In Example 8.3, $|\Theta_1| - |\Theta_0| = 1$, and in this case we saw that under $H_0$,

$$2 \log \Lambda \sim \chi^2_1$$

exactly for all $n$ in that particular case, rather than just approximately for large $n$ as the Theorem shows.

(Of often the likelihood ratio is calculated with the null hypothesis in the numerator, and so the test statistic is $-2 \log \Lambda_{\mathbf{X}}(H_1; H_0)$.)