Lecture 2. Estimation, bias, and mean squared error

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For any particular observed sample $\mathbf{x}$, our estimate is $T(\mathbf{x}) = \frac{1}{n} \sum x_i$.

We have $T(\mathbf{X}) \sim N(\mu, 1/n)$. $\square$

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[Notation note: when a parameter subscript is used with an expectation or variance, it refers to the parameter that is being conditioned on. i.e. the expectation or variance will be a function of the subscript]

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[NB: sometimes it can be preferable to have a biased estimator with a low variance - this is sometimes known as the 'bias-variance tradeoff'.]

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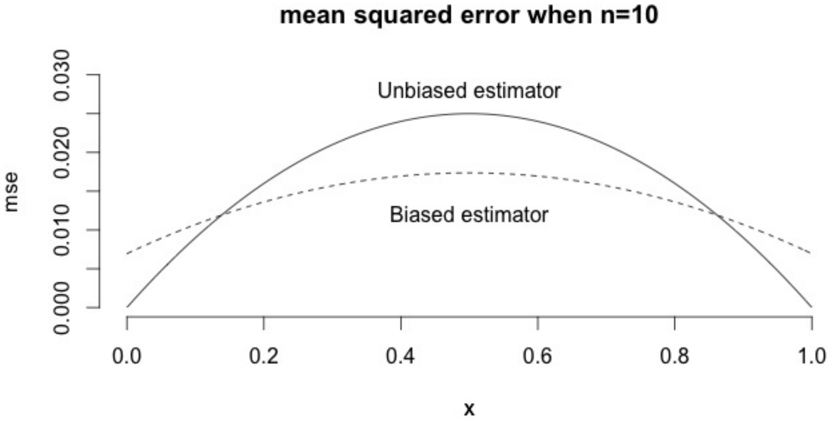
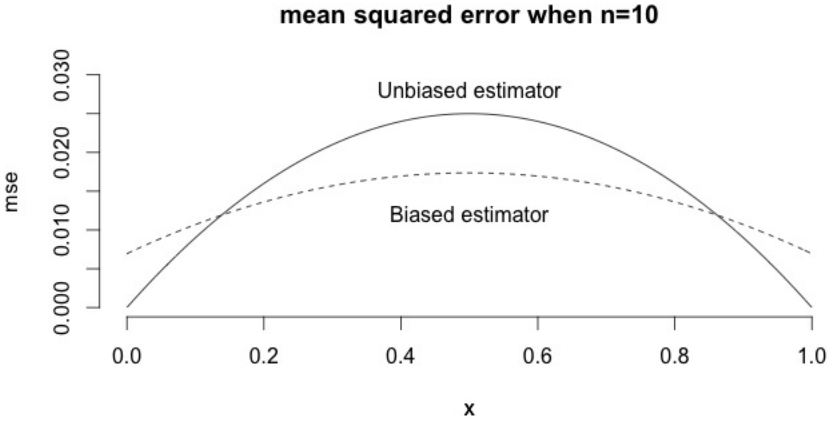

mean squared error when n=10
Unbiased estimator
Biased estimator
mse
0.000
0.010
0.020
0.030
0.0
0.2
0.4
0.6
0.8
1.0
x



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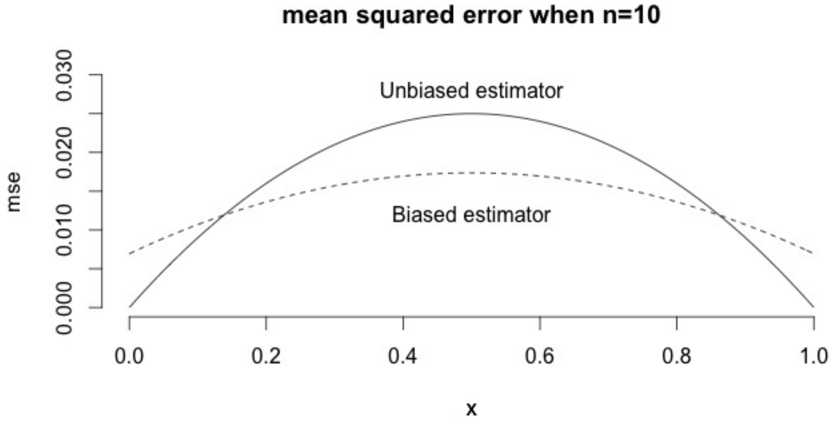

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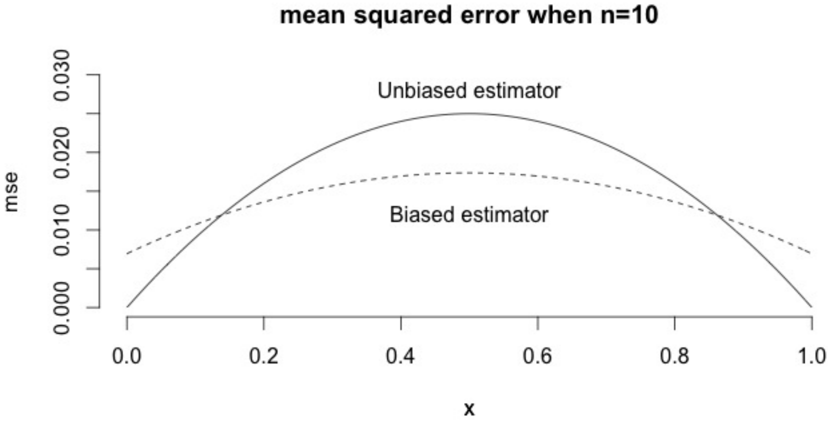

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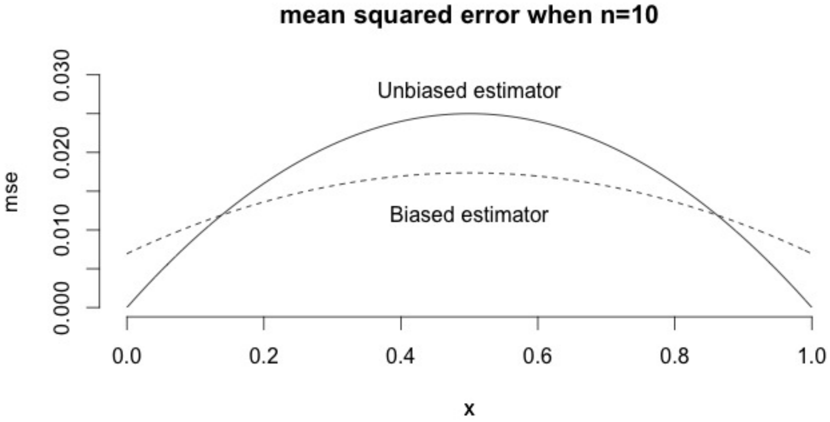

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Will see more of this when we come to Bayesian methods,.

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This is not sensible.