Variation independence

- $X, Y$ random variables, joint probability distribution parameterised by $\theta \in \Theta$.

- Joint density:
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- Where $\theta \approx (\theta_X, \theta_{Y|X}) \approx (\theta_Y, \theta_{X|Y})$.

- Desire variation independence in both forms:
  $\Theta \approx \Theta_X \times \Theta_{Y|X} \approx \Theta_Y \times \Theta_{X|Y}$

- Called strong meta Markov property.

- Maximum likelihood estimators can be derived for each factor independently.

- Profile likelihood $\propto$ marginal likelihood.
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  • $X =$ discrete covariates (age band, sex, etc.)
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Assume $(X,Y)$ arbitrary multinomial distribution with proportional odds constraint:

\[
\log \frac{p_{XY}(x,1)}{p_{XY}(x,0)} = \alpha + X^\top \beta
\]

where $\beta$ parameter of interest.

Prospective: logistic regression (easy)

Case-control: lots of parameters (complex)

Model is strong meta Markov ⇒ Likelihood function for $\beta$ same shape ⇒ Case-control MLE $\hat{\beta}$ can be found by logistic regression.
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