Brownian motion with variable drift: 0-1 laws, hitting probabilities and Hausdorff dimension

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Abstract

By the Cameron–Martin theorem, if a function f is in the Dirichlet space D, then $B + f$ has the same a.s. properties as standard Brownian motion, B. In this paper we examine properties of $B + f$ when $f \notin D$. We start by establishing a general 0-1 law, which in particular implies that for any fixed f, the Hausdorff dimension of the image and the graph of $B + f$ are constants a.s. (This 0-1 law applies to any Lévy process.) Then we show that if the function f is Hölder($\frac{1}{2}$), then $B + f$ is *intersection equivalent* to B. Moreover, $B + f$ has double points a.s. in dimensions $d \leq 3$, while in $d \geq 4$ it does not. We also give examples of functions which are Hölder with exponent less than $\frac{1}{2}$, that yield double points in dimensions greater than 4. Finally, we show that for $d \geq 2$, the Hausdorff dimension of the image of $B + f$ is a.s. at least the maximum of 2 and the dimension of the image of f.

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1 Introduction

By the Cameron–Martin theorem, if a function f is in the Dirichlet space,

$$D[0,1] = \left\{ F \in C[0,1] : \exists f \in \mathbf{L}^2[0,1] \text{ such that } F(t) = \int_0^t f(s)ds, \forall t \in [0,1] \right\},$$

then $B + f$ has the same a.s. properties as standard Brownian motion, B. When f is not in this space, then the laws of B and $B + f$ are singular, i.e., there exists a Borel set A in $C[0,1]$ such that $\mathbb{L}_B(A) = 0$ and $\mathbb{L}_{B+f}(A^c) = 0$, where $\mathbb{L}_B$ and $\mathbb{L}_{B+f}$ are the laws of B and $B + f$, respectively. Thus, if $f \notin D[0,1]$, then there exists an almost sure property of B which is not an almost sure property of $B + f$.

In this paper we consider functions $f \notin D[0,1]$, and investigate whether some specific a.s. properties of Brownian motion transfer also to $B + f$.

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In Section 2 we establish a zero-one law for random set functions, Theorem 2.1. Here we state some special cases of this law. For a function $f : [0,1] \to \mathbb{R}^d$, denote its graph restricted to $A \subset [0,1]$ by

$$G(f)(A) = \{(t, f(t)) : t \in A\}.$$

When $A = [0,1]$, we abbreviate $G(f)([0,1])$ as $G(f)$.

For concreteness, we state the following theorem for Brownian motion. Parts (**a**) and (**c**) apply to any càdlàg process with independent increments.

Theorem 1.1. *Let $(B_t, 0 \le t \le 1)$ be a standard Brownian motion in $\mathbb{R}^d$ and let $f : [0,1] \to \mathbb{R}^d$ be a continuous function.*

(**a**) *Let A be a closed set in $[0,1]$. Then $\mathbb{P}(\mathcal{L}(B+f)(A) > 0) \in \{0,1\}$, where $\mathcal{L}$ stands for Lebesgue measure and $(B+f)(A)$ is the image of A under $B+f$.*

(**b**) *Let $A \subset [0,1]$ be closed. Then $\mathbb{P}(((B+f)(A))^\circ \neq \varnothing) \in \{0,1\}$, where D° stands for the interior of D.*

(**c**) $\dim(B+f)[0,1] = c$ *a.s. and* $\dim G(B+f)[0,1] = c'$ *a.s. where c and c' are two positive constants and $\dim$ is the Hausdorff dimension.*

Remark 1.2. We know that in dimensions $d \ge 2$, Brownian motion without drift has 0 volume. A natural question that arises is whether there exist functions f such that the Lebesgue measure of $(B+f)[0,1]$ is positive. Such functions have been constructed by Graversen [7]. For all $d \ge 2$ and all $\alpha < \frac{1}{d}$, Antunović, Peres and Vermesi [2] construct an α-Hölder continuous function, f, which makes the image of $B+f$ cover an open set a.s. Related results for Lévy processes are in [5].

In Section 2.2 we discuss an example of an event which seems similar to those in Theorem 1.1, yet does not follow a 0-1 law. This example answers a question of Itai Benjamini (personal communication).

Before stating the other results we obtain in this paper, we first recall some definitions.

Definition 1.3. Let $X = (X_t, t \ge 0)$ be a stochastic process. A closed set $A \subset \mathbb{R}^d$ is called **nonpolar** for X, if $\mathbb{P}_x(X \text{ hits } A) > 0$, for all x. Otherwise, it is called **polar**.

Definition 1.4. Let B be a standard Brownian motion in 2 dimensions. We denote by B^λ, the Brownian motion killed after an independent Exponential time of parameter λ.

In Section 3 we obtain results about the hitting probabilities of $B+f$, when f is a Hölder$\left(\frac{1}{2}\right)$ continuous function. We recall the definition of Hölder(α) continuous function:

$$\exists K : \forall x, y \quad |f(x) - f(y)| \le K|x - y|^\alpha.$$

We call K the Hölder(α) constant of f. We will prove the following theorem:

Theorem 1.5. *Let $(B_t)_t$ be a standard Brownian motion in $d \ge 2$ dimensions and f a Hölder$\left(\frac{1}{2}\right)$ continuous function $f : \mathbb{R}_+ \to \mathbb{R}^d$ with Hölder constant K.*

(a)   If $d \geq 3$, then $B + f$ and B are intersection equivalent, in the sense that there exist positive constants c_1 and c_2 depending only on d and K such that for all $x \in \mathbb{R}^d$ and all closed sets $A \subset \mathbb{R}^d$, we have

$$c_1 \mathbb{P}_x(B \text{ hits } A) \leq \mathbb{P}_x(B + f \text{ hits } A) \leq c_2 \mathbb{P}_x(B \text{ hits } A),$$

where the notation $\mathbb{P}_x(B + f \text{ hits } A)$ means that $B_0 + f(0) = x$. In particular, if a closed set Λ is polar for B, then it is also polar for $B + f$.

(b)   If $d = 2$, then for any bounded open set U, there exist positive constants c_1 and c_2 depending on K, on U and λ such that for all $x \in U$ and all closed sets $A \subset U$, we have that

$$c_1 \mathbb{P}_x(B^\lambda \text{ hits } A) \leq \mathbb{P}_x((B + f)^\lambda \text{ hits } A) \leq c_2 \mathbb{P}_x(B^\lambda \text{ hits } A).$$

Also if a closed set A is nonpolar for B, then it is also nonpolar for $B + f$ and $\mathbb{P}_x(B + f \text{ hits } A) = 1$, for all x. Finally, if A, a closed set, is polar for B, then it is also polar for $B + f$.

Next, in Section 4, using the "intersection equivalence" given in Theorem 1.5, we show that when the function f is again a Hölder$(\frac{1}{2})$ continuous function, then double points for $B + f$ exist only in dimension $d \leq 3$, just like in the Brownian motion case. We will prove the following theorem:

Theorem 1.6. *Let $(B_t, 0 \leq t \leq 1)$ be a standard Brownian motion in d dimensions and let f be a Hölder$\left(\frac{1}{2}\right)$ function.*

(a) *If $d \leq 3$, then, almost surely, $B + f$ has double points.*

(b) *If $d \geq 4$, then, almost surely, $B + f$ has no double points.*

In Theorem 4.2 we show that in dimension $d \geq 4$, there exist $f \notin$ Hölder$(\frac{1}{2})$ such that $B + f$ has double points.

Remark 1.7. 1. The space of Hölder(α) continuous functions functions is much larger than the Dirichlet space, $D[0, 1]$. Indeed, for any $\alpha \in (0, 1)$, most Hölder(α) continuous functions (in the sense of Baire category) are nowhere differentiable.
2. The value $\alpha = 1/2$ is also the critical Hölder exponent for other properties of Brownian motion with drift, such as positive area in 2 dimensions (see [7]) and isolated zeros in 1 dimension (see [1]).

Finally, in Section 5, we study the Hausdorff dimension of the image and graph of $B + f$, when f is a continuous function.

From Theorem 1.1 (part c), we have that the Hausdorff dimension of the image and the graph are constants a.s. In Section 5 we obtain upper and lower bounds for these constants.

Recall from McKean's theorem (see for instance [12, Theorem 4.33]) that almost surely

$$\dim B(A) = (2 \dim A) \wedge d.$$

In the following Theorem we show that adding a continuous drift cannot decrease the Hausdorff dimension of the image.

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Theorem 1.8. *Let $f : [0,1] \to \mathbb{R}^d$, $d \geq 1$, be a continuous function and let $(B_t, 0 \leq t \leq 1)$ be a standard Brownian motion in d dimensions. Let A be a closed subset of $[0,1]$. Then*

$$\dim(B+f)(A) \geq \max\{(2\dim A) \wedge d, \dim f(A)\}.$$

Our next result concerns the dimension of the graph.

Theorem 1.9. *Let f and B be as in Theorem 1.8.*

(a) *If $d = 1$, then $\dim G(B+f) \geq \max\{\frac{3}{2}, \dim G(f)\}$.*

(b) *If $d \geq 2$, then $\dim G(B+f) \geq \max\{2, \dim G(f)\}$.*

2   0-1 laws

In this Section we first prove the general zero one law announced in the Introduction and then apply it to give the proof of Theorem 1.1.

2.1   The theorems

Theorem 2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let Ψ be a random set function, i.e. $\Psi : \Omega \times \{$ closed sets in $[0,1]\} \to [0,\infty]$, such that $\Psi(A)$ is a random variable for each closed set $A \subset [0,1]$. Suppose that Ψ satisfies:*

1. *For $I_1, \cdots, I_j$ disjoint closed intervals, $\Psi(I_1), \cdots, \Psi(I_j)$ are independent random variables.*

2. *$\mathbb{P}\left(\forall x \in [0,1] : \Psi\{x\} = 0\right) = 1$.*

3. *$\Psi(A) \leq \Psi(B)$ a.s., whenever $A \subset B$ and they are both closed.*

4. *With probability 1 $\Psi(\omega, \cdot)$ satisfies the following: If $(A_i)_i$ is a sequence of closed sets such that $\Psi(A_j) = 0, \forall j$, and $\bigcup_{j=1}^{\infty} A_j$ is a closed set, then $\Psi\left(\bigcup_{j=1}^{\infty} A_j\right) = 0$.*

Then $\mathbb{P}\left(\Psi\left([0,1]\right) > 0\right) \in \{0,1\}$.

Proof. Let $\mathcal{D}_n$ denote the set of dyadic intervals of level n, i.e.

$$\mathcal{D}_n = \left\{ \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right] : k = 1, \cdots, 2^n \right\}.$$

We declare an interval $I \in \mathcal{D}_n$ to be **good** for the particular ω, if $\Psi(I) > 0$. Define $p_I = \mathbb{P}(\Psi(I) > 0)$. Let Z_n denote the number of good intervals of level n, i.e.

$$Z_n = \sum_{I \in \mathcal{D}_n} \mathbf{1}(\Psi(I) > 0).$$

Then $\mathbb{E}(Z_n) = \sum_{I \in \mathcal{D}_n} p_I$. From condition 4, we see that Z_n is an increasing sequence, thus $\mathbb{E}(Z_n)$ is increasing and hence converges. There are two possibilities for the limit, it is either infinite or finite.

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- If $\mathbb{E}(Z_n) \uparrow \infty$, i.e. $\sum_{I \in \mathcal{D}_n} p_I \uparrow \infty$ as $n \to \infty$, then we define

$$\mathcal{D}'_n = \left\{ \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right] : k = 1, \cdots, 2^n, k \text{ is odd} \right\}$$

and

$$\mathcal{D}''_n = \left\{ \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right] : k = 1, \cdots, 2^n, k \text{ is even} \right\}.$$

Since $\sum_{I \in \mathcal{D}_n} p_I \uparrow \infty$, at least one of the two sequences $\sum_{I \in \mathcal{D}'_n} p_I$ or $\sum_{I \in \mathcal{D}''_n} p_I$ converges to infinity.

Assuming wlog that $\sum_{I \in \mathcal{D}'_n} p_I \to \infty$, we have that

$$\mathbb{P}(\Psi([0,1]) > 0) = \mathbb{P} \left(\Psi \left(\bigcup_{I \in \mathcal{D}_n} I \right) > 0 \right) = \mathbb{P}(\exists I \in \mathcal{D}_n \text{ s.t. } \Psi(I) > 0)$$

$$\geq \mathbb{P}(\exists I \in \mathcal{D}'_n \text{ s.t. } \Psi(I) > 0) = 1 - \prod_{I \in \mathcal{D}'_n} (1 - p_I),$$

where to get the second equality we used conditions 3 and 4 and to get the last equality we used the independence assumption 1, since the intervals in $\mathcal{D}'_n$ are disjoint. But $\sum_{I \in \mathcal{D}'_n} p_I \to \infty$, so we get that $\mathbb{P}(\Psi([0,1]) > 0) = 1$.

- If $E(Z_n) \uparrow C$, where C is a finite positive constant, then we will show that $\mathbb{P}(\Psi([0,1]) > 0) = 0$. We now declare a point of $[0,1]$ to be **good** for a particular ω, if all dyadic intervals that contain it are good.

It is easy to see that if for a realization ω, there are at least k good points, then there exists n_0 such that $Z_{n_0}(\omega) \geq k$. So, if there is an infinite number of good points, then $Z_n \to \infty$ as $n \to \infty$. Let $S(\omega)$ denote the set of good points for the realization ω. Then we have that

$$\mathbb{E}(Z_n) \geq \mathbb{E}(Z_n \mathbf{1}(|S| = \infty)).$$

If $\mathbb{P}(|S| = \infty) > 0$, then from the discussion above and using monotone convergence we get that $E(Z_n) \uparrow \infty$, which contradicts our assumption that $E(Z_n) \uparrow C$. Hence a.s. there are only finitely many good points.

So, for a.a. ω, we have that $|S(\omega)| < \infty$. Take such an ω. Then we write $\mathcal{D} = \cup_n \mathcal{D}_n$ and we have

$$[0,1] = S(\omega) \bigcup \bigcup_{\substack{I \in \mathcal{D} \\ \Psi(I)=0}} I,$$

because if an interval I has no good points, then $\Psi(I) = 0$. Indeed, assuming the contrary, i.e. that $\Psi(I) > 0$, then we would get a decreasing sequence I_n of closed intervals with $\Psi(I_n) > 0$ and of lengths converging to 0. But since the space is complete, the intersection of these closed intervals would have to be non-empty, and hence we would obtain a good point.

Thus, since by condition 2 we have that $\Psi(S(\omega)) = 0$, then using also condition 4 we get that $\Psi([0,1]) = 0$. Hence we showed that in this case $\Psi([0,1]) = 0$ a.s.

$\square$

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Remark 2.2. Let μ be the counting measure for the Poisson process on $[0,1]$. Then it satisfies all conditions of Theorem 2.1, except for (2), yet there is no 0-1 law.

Corollary 2.3. *Let X be a continuous process on $[0,1]$ that satisfies $\mathbb{P}(A|B) \geq c\mathbb{P}(A)$, for all $A \in \sigma\{X(s) - X(t) : s \geq t\}$ and $B \in \sigma\{X(s) : s \leq t\}$ for some constant $c < 1$, and let f be a continuous function. Then it satisfies again a 0-1 law, namely $\mathbb{P}(\mathcal{L}((X + f)[0,1]) > 0) \in \{0,1\}$.*

Proof. We divide the unit interval in dyadic subintervals in the same way as before and declare an interval I good if $\mathcal{L}((X + f)(I)) > 0$ and let Z_n be the total number of good intervals of level n. Then there are again two possibilities, either $E(Z_n) \to \infty$ or $E(Z_n) \to C$, where C is a finite positive constant. In the second case, everything follows in the same way as in the proof of Theorem 2.1. Now, if $E(Z_n) \to \infty$, i.e. in the same notation as above $\sum_{I \in \mathcal{D}_n} p_I \to \infty$, then we have

$$\mathbb{P}(\mathcal{L}((X + f)[0,1]) > 0) = \mathbb{P}(\exists I \in \mathcal{D}_n : \mathcal{L}((X + f)(I)) > 0)$$
$$= 1 - \mathbb{P}(\forall I \in \mathcal{D}_n, \mathcal{L}((X + f)(I)) = 0).$$

But $\mathbb{P}(\forall I \in \mathcal{D}_n, \mathcal{L}((X + f)(I)) = 0) \leq \prod_{I \in \mathcal{D}_n}(1 - cp_I)$, using the assumption on the conditional probabilities. But $\prod_{I \in \mathcal{D}_n}(1 - cp_I) \to 0$ as $n \to \infty$, since $\sum_{I \in \mathcal{D}_n} p_I \to \infty$, hence we get that

$$\mathbb{P}(\mathcal{L}((X + f)[0,1]) > 0) = 1.$$

$\square$

Using Theorem 2.1, we can easily deduce Theorem 1.1 stated in the Introduction.

Proof of Theorem 1.1. For part **(a)** we define $\Psi(I) = \mathcal{L}(B + f)(A \cap I)$ and then we use Theorem 2.1.

For part **(b)** we define $\Psi(I) = \mathbf{1}(((B + f)(A \cap I))^o \neq \varnothing)$. The only condition that requires some justification is 4.
Let A_i be a sequence of closed sets with closed union and such that $((B + f)(A_i))^o = \varnothing, \forall i$. Then $((B + f)(\cup A_i))^o = (\cup(B + f)(A_i))^o$, and since the sets A_i are closed and $B + f$ is continuous, then $\forall i, (B + f)(A_i)$ is also a closed set. From Baire's theorem we get that the interior of $(\cup(B + f)(A_i))$ is empty, since otherwise one of the sets would have nonempty interior.

To prove part **(c)**, let $\Psi(I) = \mathbf{1}(\dim(B + f)(I) > c)$. Then it is easy to see that Ψ satisfies conditions 1, 2 and 3. Finally condition 4 follows from the countable stability property of Hausdorff dimension. $\square$

Remark 2.4. Note that it is crucial that the function f in Theorem 1.1 be deterministic. If f is an adapted continuous function, then the 0-1 law could fail. Here is an example: Let $B_t = (B_t^1, B_t^2)$ be a two dimensional Brownian motion started from the origin. Let A_t be a process defined as follows: for $t \leq \frac{1}{2}$, define $A_t = 0$ and for $t > \frac{1}{2}$, if $B_{\frac{1}{2}}^1 > 0$, then let A_t

be the modified Hilbert curve as defined in [2], otherwise set it to 0. In [2] it is proven that $B + A$ is space filling. Then we have that

$$\mathbb{P}(\mathcal{L}(B+A)[0,1] > 0) = \mathbb{P}(B_{\frac{1}{2}}^1 > 0) = \frac{1}{2}.$$

We now generalize Theorem 2.1 to any Polish space X.

Theorem 2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a Polish space, i.e. a complete separable metric space. Let Ψ be a random set function, i.e. $\Psi : \Omega \times \{compact\ sets\ of\ X\} \to [0, \infty]$ such that for each compact subset A of X, we have that $\Psi(A)$ is a random variable and it satisfies the following:*

1. *$\Psi(I_1), \cdots, \Psi(I_j)$ are independent random variables for $I_1, \cdots, I_j$ disjoint closed balls,*

2. *$\mathbb{P}(\forall x \in X : \Psi\{x\} = 0) = 1$.*

3. *$\Psi(A) \leq \Psi(B)$ a.s., whenever $A \subset B$ and they are both compact and*

4. *With probability 1 $\Psi(\omega, \cdot)$ satisfies: if $(A_i)_i$ is a sequence of compact sets such that $\Psi(A_j) = 0 \quad \forall j$, and $\bigcup_{j=1}^{\infty} A_j$ is a compact set, then $\Psi\left(\bigcup_{j=1}^{\infty} A_j\right) = 0$.*

Then we have that $\mathbb{P}(\Psi(X) > 0) \in \{0, 1\}$.

Proof. Since X is separable there exists a countable dense subset, which we denote by D. Let $\Gamma = \{\bar{B}(x, r) : x \in D, r \in \mathbb{Q}\}$. For a particular ω, we declare a closed ball B to be **good** if $\Psi(B) > 0$. A subset Λ of Γ is of type 1 if it contains only disjoint sets. For such a Λ we define Z_Λ to be the number of good balls in Λ. Then $E(Z_\Lambda) = \sum_{B \in \Lambda} \mathbb{P}(\Psi(B) > 0)$ and there are two possibilities.

- If $\sup_{\Lambda \subset \Gamma:\ \text{type 1}} \sum_{B \in \Lambda} \mathbb{P}(\Psi(B) > 0) = \infty$, then if Λ is of type 1, we get

$$\mathbb{P}(\Psi(X) > 0) \geq \mathbb{P}(\exists B \in \Lambda : \Psi(B) > 0) = 1 - \prod_{B \in \Lambda} (1 - \mathbb{P}(\Psi(B) > 0))$$

$$\geq 1 - \exp(-\sum_{B \in \Lambda} \mathbb{P}(\Psi(B) > 0)).$$

Taking sup over all $\Lambda \subset \Gamma$ of type 1, we deduce that $\mathbb{P}(\Psi(X) > 0) = 1$.

- If $\sup_{\Lambda \subset \Gamma:\ \text{type 1}} \sum_{B \in \Lambda} \mathbb{P}(\Psi(B) > 0) < \infty$, then again for an $\omega \in \Omega$, we declare a point of X to be good, if every ball in Γ that contains it is good. Note that if there are at least k good points, then there must exist a family Λ of type 1, such that $Z_\Lambda \geq k$. Let $S(\omega)$ denote the set of good points. On the event that $|S(\omega)| = \infty$, we can find a sequence $(\Lambda_n)_n$ such that $Z(\Lambda_n) \uparrow \infty$. Thus, if $\mathbb{P}(|S| = \infty) > 0$, then by monotone convergence we get that $\sup_{\Lambda \subset \Gamma:\ \text{type 1}} \sum_{B \in \Lambda} \mathbb{P}(\Psi(B) > 0) = \infty$, contradiction. Hence there is only a finite number of good points, S, and we now decompose X as

$$X = S \bigcup \bigcup_{B \in \Gamma:\, B \cap S = \varnothing} B.$$

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For any B that does not contain any good points we have that $\Psi(B) = 0$, since if $\Psi(B) > 0$, then we could cover B by a countable number of balls of radius $\frac{1}{2}$ and one of them would have $\Psi(B_1) > 0$ and continuing in the same way we would obtain a decreasing sequence of closed balls of radii tending to 0 and thus since the space is complete, this intersection would have to be nonempty, hence we would obtain a good point.

Finally using conditions 2 and 3 we get that a.s. $\Psi(X) = 0$.

$\square$

2.2 An event of intermediate probability

Here we present an event that seems similar to those discussed in Theorem 1.1, yet does not obey a 0-1 law. This example was mentioned in [12, Exercise 9.8], but there the proof is only sketched. We include it here with more details.

We first recall the definition of the capacity of a set.

Definition 2.6. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a kernel and A a Borel set in $\mathbb{R}^d$. The K-energy of a measure μ is defined to be

$$\mathcal{E}_K(\mu) = \int \int K(x, y) d\mu(x) d\mu(y)$$

and the K-capacity of A is defined as

$$\mathrm{Cap}_K(A) = [\inf\{\mathcal{E}_K(\mu) : \mu \text{ a probability measure on } A\}]^{-1}.$$

When the kernel has the form $K(x, y) = |x - y|^{-\alpha}$, then we write $\mathcal{E}_\alpha(\mu)$ for $\mathcal{E}_K(\mu)$ and $\mathrm{Cap}_\alpha(A)$ for $\mathrm{Cap}_K(A)$ and we refer to them as the α-energy of μ and the Riesz α-capacity of A respectively.

We recall the following theorem (its proof can be found in [12, Theorem 4.32] for instance), which gives the connection between the Hausdorff dimension and the Riesz α-capacity, because it will be used extensively in this paper.

Theorem 2.7 (Frostman). *For any closed set $A \subset \mathbb{R}^d$,*

$$\dim A = \sup\{\alpha : \mathrm{Cap}_\alpha(A) > 0\}.$$

Example 2.8. *Let $(B_t, 0 \leq t \leq 3)$ be a standard Brownian motion in one dimension. Let A be a closed subset of $[0, 3]$. Then we will show that it is not always true that $\mathbb{P}(B \text{ is 1-1 on } A) \in \{0, 1\}$.*

Proof. It is clear that if A is any closed interval, then the above probability is 0.

We are going to use the following equivalence ([9, Theorem 1.1]): for any two disjoint closed sets Λ_0 and Λ_1,

$$\mathbb{P}(B(\Lambda_0) \cap B(\Lambda_1) \neq \varnothing) > 0 \Leftrightarrow \mathrm{Cap}_{\frac{1}{2}}(\Lambda_0 \times \Lambda_1) > 0. \tag{2.1}$$

Let
$$A_0 = \{\sum_{n=1}^{\infty} \frac{x_n}{2^n} : x_n \in \{0,1\} \text{ and } \forall k, \forall n \in ((2k)!, (2k+1)!], x_n = 0\}$$
be a closed subset of $[0,1]$ and let
$$A_1 = \{2 + \sum_{n=1}^{\infty} \frac{x_n}{2^n} : x_n \in \{0,1\} \text{ and } \forall k, \forall n \in ((2k-1)!, (2k)!], x_n = 0\}$$
be a closed subset of $[2,3]$.

We will show that $\mathbb{P}(B \text{ is 1-1 on } A_0 \cup A_1) \notin \{0,1\}$. This probability is equal to
$$\mathbb{P}(B(A_0) \cap B(A_1) = \varnothing, B \text{ is 1-1 on } A_0, B \text{ is 1-1 on } A_1).$$

We have that $\mathbb{P}(B(A_0) \cap B(A_1) \neq \varnothing)$ is strictly smaller than 1, since $A_0 \subset [0,1]$ and $A_1 \subset [2,3]$ and with positive probability the images of $[0,1]$ and $[2,3]$ under Brownian motion are disjoint.

It is easy to see that $\dim(A_0) = 0$. Indeed, for each k, we can cover A_0 by at most $2^{(2k)!}$ dyadic intervals of length $2^{-(2k+1)!}$. Hence, for all k,
$$\mathcal{H}_{\infty}^{\alpha}(A_0) \leq 2^{(2k)!}(2^{-(2k+1)!})^{\alpha} \to 0, \text{ as } k \to \infty.$$
The same argument also gives that $\dim(A_0 \times A_0) = 0$ and $\dim(A_1 \times A_1) = 0$.

Note that $A_0 + A_1 = \{x + y : x \in A_0, y \in A_1\} = [2,3]$ and since the mapping $f : A_0 \times A_1 \to [2,3]$ given by $f(x,y) = x + y$ is Lipschitz with constant 1, we get that $\dim(A_0 \times A_1) \geq 1$, and hence by Theorem 2.7, $\text{Cap}_{\frac{1}{2}}(A_0 \times A_1) > 0$, so from (2.1) we deduce that
$$\mathbb{P}(B(A_0) \cap B(A_1) \neq \varnothing) > 0.$$

We now want to show that $\mathbb{P}(B \text{ is 1-1 on } A_0) = 1$ and similarly for A_1.
Let $\Delta = \{(x,x) : x \in \mathbb{R}\}$ and note that we can write
$$A_0 \times A_0 \setminus \Delta = \bigcup_{\substack{a,b,c,d \in \mathbb{Q} \\ [a,b] \cap [c,d] = \varnothing}} ([a,b] \cap A_0) \times ([c,d] \cap A_0).$$

We now have
$$\mathbb{P}(B \text{ not 1-1 on } A_0) = \mathbb{P}(\exists \quad (s,t) \in A_0 \times A_0 \setminus \Delta : B(s) = B(t))$$
$$\leq \sum_{\substack{a,b,c,d \in \mathbb{Q} \\ [a,b] \cap [c,d] = \varnothing}} \mathbb{P}(B([a,b] \cap A_0) \cap B([c,d] \cap A_0) \neq \varnothing).$$

But $[a,b] \cap A_0$ and $[c,d] \cap A_0$ are disjoint and closed sets, so from (2.1) we get that
$$\mathbb{P}(B([a,b] \cap A_0) \cap B([c,d] \cap A_0) \neq \varnothing) = 0,$$
since $\dim(([a,b] \cap A_0) \times ([c,d] \cap A_0)) = 0$.

Therefore we deduce that $\mathbb{P}(B \text{ is 1-1 on } A_0 \cup A_1) = \mathbb{P}(B(A_0) \cap B(A_1) = \varnothing)$, and hence strictly between 0 and 1.

$\square$

3 Hitting probabilities

In this section, we are going to give the proof of Theorem 1.5. We will need a preliminary Proposition, given just below, which compares the Green kernels for B and $B + f$. Recall that the transition density of the Brownian motion in d dimensions is

$$p(t, x, y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}}$$

and the corresponding Green kernel is

$$G(x, y) = \int_0^\infty p(t, x, y) \, dt.$$

Similarly the transition density for $B + f$ is given by

$$\tilde{p}(t, x, y) = p(t, x - f(0), y - f(t))$$

and the corresponding Green kernel is given by $\tilde{G}(x, y) = \int_0^\infty \tilde{p}(t, x, y) \, dt$.

In dimension 2 we consider the Green kernel for the killed process $G_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) \, dt$ and define $\tilde{G}_\lambda$ analogously.

Proposition 3.1. *Let $(B_t)_t$ be a transient Brownian motion in $d \geq 2$ dimensions. In dimension 2 we kill the Brownian motion after an independent Exponential time of parameter λ. Let f be a Hölder$(\frac{1}{2})$ continuous deterministic function $f : \mathbb{R}_+ \to \mathbb{R}^d$ with Hölder constant K and let $\tilde{G}(x, y)$ be the Green kernel of the process $B + f$.*

(a) *If $d \geq 3$, then there exist positive constants c_1 and c_2 depending only on d and K, such that for all $x, y \in \mathbb{R}^d$, we have $c_1 G(x, y) \leq \tilde{G}(x, y) \leq c_2 G(x, y)$.*

(b) *If $d = 2$, then for all $C > 0$, there exist positive constants c_1 and c_2, depending on C, K and λ, such that for all $x, y \in \mathbb{R}^2$ such that $|x - y| \leq C$, we have that $c_1 G_\lambda(x, y) \leq \tilde{G}_\lambda(x, y) \leq c_2 G_\lambda(x, y)$.*

Proof of (a). Since $|x - y + f(t) - f(0)| \geq ||x - y| - |f(t) - f(0)||$, we have that

$$\tilde{G}(x, y) \leq \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}} e^{\frac{|x-y||f(t)-f(0)|}{t}} e^{-\frac{|f(t)-f(0)|^2}{2t}} \, dt,$$

and since f is Hölder$(\frac{1}{2})$ we have that $|f(t) - f(0)| \leq K\sqrt{t}$, for all t for a positive constant K, and setting $r = |x - y|$ we get

$$\tilde{G}(x, y) \leq \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{2t}} e^{\frac{Kr}{\sqrt{t}}} \, dt \leq \int_0^{t_0} \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{4t}} \, dt + c' \int_{t_0}^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{2t}} \, dt,$$

where $t_0 = cr^2$ and c is a sufficiently small constant. We also have that

$$\int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{2t}} \, dt = c(d) r^{2-d},$$

(see for instance [12, Theorem 3.33]). So,

$$\int_0^{t_0} \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{4t}} dt \le \int_0^{\infty} \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{4t}} dt = c(d) \left(\frac{r}{\sqrt{2}} \right)^{2-d},$$

thus there exists a uniform positive constant c_1 such that for all x and y

$$\tilde{G}(x,y) \le c_1 G(x,y).$$

For the lower bound, using again the Hölder continuity assumption on f, we have

$$\tilde{G}(x,y) \ge \int_0^{\infty} \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{2t}} e^{-\frac{|f(t)-f(0)|^2}{2t}} e^{-\frac{r|f(t)-f(0)|}{t}} dt \ge \tilde{c} \int_0^{\infty} \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{2t}} e^{-\frac{Kr}{\sqrt{t}}} dt \ge c_2 G(x,y),$$

for a positive constant c_2, uniform over all x and y. The last inequality follows in the same way as the upper bound above. $\qquad\square$

Proof of (b). Let x and y satisfy $r = |x-y| \le C$, for $C > 0$. Then for $t_0 = \alpha r^2$, for α a sufficiently small constant, we have that

$$\tilde{G}_{\lambda}(x,y) \le \int_0^{\infty} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} e^{\frac{Kr}{\sqrt{t}}} e^{-\lambda t} dt \le \int_0^{t_0} \frac{1}{2\pi t} e^{-\frac{r^2}{4t}} e^{-\lambda t} dt + e^{K/\sqrt{\alpha}} \int_{t_0}^{\infty} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} e^{-\lambda t} dt$$

$$= \int_0^{2t_0} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} e^{-\lambda t} e^{\frac{\lambda t}{2}} dt + e^{K/\sqrt{\alpha}} \int_{t_0}^{\infty} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} e^{-\lambda t} dt$$

$$\le e^{\lambda C^2 \alpha} \int_0^{2t_0} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} e^{-\lambda t} dt + e^{K/\sqrt{\alpha}} \int_{t_0}^{\infty} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} e^{-\lambda t} dt \le c_1 G_{\lambda}(x,y),$$

The lower bound follows similarly. $\qquad\square$

Proof of Theorem 1.5 (a). From [3, Proposition 1.1] or [12, Theorem 8.24] we have that for any transient Brownian motion

$$\frac{1}{2} \text{Cap}_M(A) \le \mathbb{P}_{x_0}(B \text{ hits } A) \le \text{Cap}_M(A), \tag{3.1}$$

where M is the Martin kernel, defined by $M(x,y) = \frac{G(x,y)}{G(x_0,y)}$, and A is a closed set. To prove the theorem, it suffices to establish that there exist positive constants c_1 and c_2 depending only on the Hölder constant of f such that for all starting points x_0 and all closed sets A

$$c_1 \text{Cap}_M(A) \le \mathbb{P}_{x_0}(B + f \text{ hits } A) \le c_2 \text{Cap}_M(A). \tag{3.2}$$

The proof of Theorem 8.24 in [12] can be adapted to show this; we will give the details for this adaptation for the upper bound on the hitting probability.

Let $\tau = \inf\{t > 0 : B_t + f(t) \in A\}$ and let ν be the distribution of $B_\tau + f(\tau)$. The total mass of ν is $\nu(A) = \mathbb{P}_{x_0}(\tau < \infty)$. From the definition of the Green kernel $\tilde{G}$ for $B + f$ we have that for any y

$$\mathbb{E} \int_0^{\infty} \mathbf{1}(|B_t + f(t) - y| < \varepsilon) \, dt = \int_{\mathcal{B}(y,\varepsilon)} \tilde{G}(x_0, z) \, dz. \tag{3.3}$$

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Using Proposition 3.1, we obtain that there exist constants c and c' that depend only on the Hölder constant of f such that for all x_0 and z

$$cG(x_0, z) \leq \tilde{G}(x_0, z) \leq c'G(x_0, z). \tag{3.4}$$

Integrating over all t the inequality

$$\mathbb{P}_{x_0}(|B_t + f(t) - y| < \varepsilon) \geq \mathbb{P}_{x_0}(|B_t + f(t) - y| < \varepsilon, \tau \leq t),$$

we get

$$\mathbb{E}_{x_0} \int_0^\infty \mathbf{1}(|B_t + f(t) - y| < \varepsilon) \, dt \geq \mathbb{E}_{x_0} \int_0^\infty \mathbf{1}(B_{t+\tau} + f(t+\tau) - y| < \varepsilon) \, dt$$

$$= \mathbb{E}_{x_0} \mathbb{E}_{x_0} \left(\int_0^\infty \mathbf{1}(|B_{t+\tau} + f(t+\tau)| < \varepsilon) \, dt | \mathcal{F}_\tau \right) = \mathbb{E}_{x_0} \int_A \int_{\mathcal{B}(y,\varepsilon)} G_\tau(x, z) \, dz \, d\nu(x),$$

where G_τ is the Green kernel for the process $B_{t+\tau} + f(t+\tau)$, given by

$$G_\tau(x, y) = \int_0^\infty p(t, x - f(\tau), y - f(t+\tau)) \, dt.$$

Given $\mathcal{F}_\tau$, by the strong Markov property $B_{t+\tau} - B_\tau$ is a standard Brownian motion independent of the past and $B_\tau + f(t+\tau)$ is a Hölder(1/2) function, independent of $B_{t+\tau} - B_\tau$, with the same Hölder constant as f. Therefore, given $\mathcal{F}_\tau$, for all x and z we have

$$cG(x, z) \leq G_\tau(x, z) \leq c'G(x, z),$$

for the same constants c and c' appearing in (3.4), since they only depend on the Hölder constant of f.

We thus obtain

$$\mathbb{E}_{x_0} \int_0^\infty \mathbf{1}(|B_t + f(t) - y| < \varepsilon) \, dt \geq c \int_A \int_{\mathcal{B}(y,\varepsilon)} G(x, z) \, dz \, d\nu(x)$$

and combining that with (3.3) and using (3.4) we deduce that

$$c' \int_{\mathcal{B}(y,\varepsilon)} G(x_0, z) \, dz \geq c \int_A \int_{\mathcal{B}(y,\varepsilon)} G(x, z) \, dz \, d\nu(x).$$

Dividing through by $\mathcal{L}(\mathcal{B}(0, \varepsilon))$ and letting $\varepsilon \to 0$ we obtain

$$c'G(x_0, y) \geq c \int_A G(x, y) \, d\nu(x),$$

and hence, $\nu(A) \leq \frac{c'}{c} \mathrm{Cap}_M(A)$. $\qquad \square$

It is a classical fact that Brownian motion in 2 dimensions is neighborhood recurrent. In the following lemma we will prove that the same is true for $B + f$, if f is a Hölder(1/2) continuous function.

Lemma 3.2. *Let f be a Hölder(1/2) continuous function, $f : \mathbb{R}_+ \to \mathbb{R}^2$ and B a standard Brownian motion in 2 dimensions. Then $B + f$ is neighborhood recurrent.*

Proof. Let D denote the unit ball in $\mathbb{R}^2$. Without loss of generality, we will show that $B + f$ hits D infinitely often almost surely. Let $w \in \mathbb{R}^2$. We will prove that

$$\mathbb{P}_w \left(\bigcap_n \{B + f \text{ hits } D \text{ after time } n\} \right) = 1.$$

The event $\bigcap_n \{B + f \text{ hits } D \text{ after time } n\}$ is a tail event for Brownian motion, and hence has probability either 0 or 1. Also,

$$\mathbb{P}_w \left(\bigcap_n \{B + f \text{ hits } D \text{ after time } n\} \right) = \lim_{n \to \infty} \mathbb{P}_w \left(B + f \text{ hits } D \text{ after time } n \right).$$

Let $T = \int_n^{n^2} \mathbf{1}(|B_t + f(t)| \leq 1) \, dt$, which is the time spent in the unit ball between n and n^2. Then

$$\mathbb{E}_w(T) = \int_n^{n^2} \mathbb{P}_w(|B_t + f(t)| \leq 1) \, dt$$

and there exist positive constants c_1 and c_2 that depend only on the Hölder constant of f such that for all $t \geq |w|^2$,

$$\frac{c_2}{t} \leq \mathbb{P}_w(|B_t + f(t)| \leq 1) \leq \frac{c_1}{t}.$$

We thus obtain that

$$\mathbb{E}_w(T) \geq c_2 \log n.$$

For the second moment of T we have

$$\mathbb{E}_w(T^2) = 2\mathbb{E}_w \int_n^{n^2} \int_s^{n^2} \mathbf{1}(|B_t + f(t)| \leq 1)\mathbf{1}(|B_s + f(s)| \leq 1) \, dt \, ds$$

$$\leq 2 \int_n^{n^2} \frac{c_1}{s} \int_s^{n^2} \left(\frac{c_1}{t-s} \wedge 1 \right) dt \, ds \leq 2 \int_n^{n^2} \frac{c_1}{s} \int_0^{n^2} \left(\frac{c_1}{u} \wedge 1 \right) du \, ds \leq c(\log n)^2,$$

for a positive constant c. Therefore, applying the second moment method to T, namely $\mathbb{P}_w(T > 0) \geq \frac{\mathbb{E}_w(T)^2}{\mathbb{E}_w(T^2)}$, we get that $\mathbb{P}_w(T > 0) \geq \frac{c_2^2}{c} > 0$, and hence

$$\mathbb{P}_w \left(B + f \text{ hits } D \text{ after time } n \right) \geq \mathbb{P}_w(T > 0) > \frac{c_2^2}{c},$$

which concludes the proof of the lemma. $\qquad\square$

Proof of Theorem 1.5 (b). From [3, Proposition 1.1] or [12, Theorem 8.24] we have that

$$\frac{1}{2}\mathrm{Cap}_{M_\lambda}(A) \leq \mathbb{P}_{x_0}(B \text{ hits } A) \leq \mathrm{Cap}_{M_\lambda}(A), \tag{3.5}$$

where M_λ is the Martin kernel for the killed Brownian motion B^λ defined by $M_\lambda(x, y) = \frac{G_\lambda(x,y)}{G_\lambda(x_0,y)}$. To prove the first part of the theorem it suffices to establish that for any bounded open set U, there exist positive constants c_1 and c_2 that depend only on U, on the Hölder constant of f and on λ such that for all $x_0 \in U$ and all closed sets $A \subset U$

$$c_1 \mathrm{Cap}_{M_\lambda}(A) \leq \mathbb{P}_{x_0}((B + f)^\lambda \text{ hits } A) \leq c_2 \mathrm{Cap}_{M_\lambda}(A). \tag{3.6}$$

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The proof of that follows in the same way as the proof of (3.2) using also Proposition 3.1(part **(b)**).

For the second part of the theorem, let A be a nonpolar set for B, i.e. $\mathbb{P}_u(B \text{ hits } A) > 0$, for all starting points u. By neighborhood recurrence of Brownian motion, we get that this probability is indeed equal to 1. Let B^λ and $(B + f)^\lambda$ denote the processes killed after an Exponential time of parameter λ independent of the Brownian motion. Let $x \in \mathbb{R}^2$. We can find a small λ such that

$$\mathbb{P}_x(B^\lambda \text{ hits } A) > c \quad \text{and} \quad \mathbb{P}_0(B^\lambda \text{ hits } A) > c,$$

for a positive constant c. Using (3.6) we deduce that also $\mathbb{P}_x((B + f)^\lambda \text{ hits } A) > 0$, and hence

$$\mathbb{P}_x(B + f \text{ hits } A) > 0.$$

We now need to show that $\mathbb{P}_x(B + f \text{ hits } A) = 1$.

The event $\{B + f \text{ hits } A \text{ i.o.}\}$ is a tail event, and hence has probability either 0 or 1. Without loss of generality we assume that A is separated from the ball of radius 2 centered at the origin. By (3.6) for a ball C of radius $\frac{1}{2}$ around 0, we can find positive constants c_3 and c_4 that depend only on C and on the Hölder constant of f, such that for all z in C we have that

$$c_3 \mathbb{P}_z(B^\lambda \text{ hits } A) \le \mathbb{P}_z((B + f)^\lambda \text{ hits } A) \le c_4 \mathbb{P}_z(B^\lambda \text{ hits } A). \tag{3.7}$$

First we will show that

$$\inf_{y \in C} \mathbb{P}_y(B^\lambda \text{ hits } A) > c_5 > 0, \tag{3.8}$$

and hence, using (3.7), we will get that

$$\inf_{y \in C} \mathbb{P}_y(B + f \text{ hits } A) > c_6 > 0. \tag{3.9}$$

To show (3.8), we will show that for all $x_0 \in C$

$$\mathbb{P}_{x_0}(B^\lambda \text{ hits } A) \ge c_7 \mathbb{P}_0(B^\lambda \text{ hits } A), \tag{3.10}$$

for a positive constant c_7.

The probability $\mathbb{P}_0(B^\lambda \text{ hits } A)$ is bounded from above by the probability that a Brownian motion without killing started from 0 hits the boundary of $\mathcal{B}(x_0, 1)$, denoted by $\partial\mathcal{B}(x_0, 1)$, where $\mathcal{B}(x_0, 1)$ is the ball of radius 1 centered at $x_0 \in C$, and then starting from the hitting point an independent Brownian motion with Exponential(λ) killing hits A. Using Poisson's formula (it can be found in [12, Theorem 3.44] for instance) we obtain

$$\mathbb{P}_0(B^\lambda \text{ hits } A) \le \int_{\partial\mathcal{B}(x_0,1)} \frac{1 - |x_0|^2}{|z|^2} \mathbb{P}_z(B^\lambda \text{ hits } A) d\varpi(z) \le 4 \int_{\partial\mathcal{B}(x_0,1)} \mathbb{P}_z(B^\lambda \text{ hits } A) d\varpi(z),$$

where ϖ stands for the uniform distribution on the sphere $\partial\mathcal{B}(x_0, 1)$ We also have

$$\mathbb{P}_{x_0}(B^\lambda \text{ hits } A) = \int_{\partial\mathcal{B}(x_0,1)} \mathbb{P}_{x_0}(T_{\partial\mathcal{B}(x_0,1)} \le T(\lambda)) \mathbb{P}_z(B^\lambda \text{ hits } A) d\varpi(z)$$

$$= c_8 \int_{\partial\mathcal{B}(x_0,1)} \mathbb{P}_z(B^\lambda \text{ hits } A) d\varpi(z) \ge \frac{c_8}{4} \mathbb{P}_0(B^\lambda \text{ hits } A),$$

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where $T(\lambda)$ is the Exponential killing time of parameter λ and $c_8 = \mathbb{P}_y(T_{\partial \mathcal{B}(y,1)} \leq T(\lambda))$, a constant independent of y.

Let $T_n = \inf\{t \geq n : B(t) + f(t) \in C\}$. By the neighborhood recurrence, Lemma 3.2, we get that $T_n < \infty$ a.s. and thus we have

$$\mathbb{P}_x(B + f \text{ hits } A \text{ after time } n) \geq \mathbb{P}_x(B + f \text{ hits } C \text{ after time } n, \text{ then hits } A)$$
$$\geq \inf_{y \in C} \mathbb{P}_y(B(t + T_n) + f(t + T_n) \text{ hits } A) = \inf_{y \in C} \mathbb{P}_y(\tilde{B}(t) + \tilde{f}(t) \text{ hits } A),$$

where $\tilde{B}(t) = B(t + T_n) - B(T_n)$ is by the strong Markov property a standard Brownian motion and $\tilde{f}(t) = f(t + T_n) - f(T_n) + y$, which is still Hölder(1/2) continuous with the same constant as f and conditioned on T_n, $\tilde{f}$ is independent of $\tilde{B}$. Hence, from (3.9), since the constant c_6 depends only on the Hölder constant of f, we finally get that

$$\mathbb{P}_x(B + f \text{ hits } A \text{ after time } n) \geq c_9 > 0,$$

therefore

$$\mathbb{P}_x(B + f \text{ hits } A \text{ i.o.}) = 1.$$

The last part of the Theorem, namely that polar sets for Brownian motion are also polar sets for $B + f$ follows easily from equation (3.7), and this completes the proof of the theorem. $\quad\square$

4 Double points

In this section, we give the proof of Theorem 1.6, which was stated in the Introduction. Some of the proofs in this section are similar to those for Brownian motion without drift.

Theorem 4.1. *Let B_1 and B_2 be two independent Brownian motions in d dimensions and let f_1 and f_2 be two Hölder($\frac{1}{2}$) continuous deterministic functions $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}^d$ with the same Hölder constant, K.*

(a) *If $d \geq 4$, then, almost surely, $(B_1 + f_1)[0, \infty)$ and $(B_2 + f_2)[0, \infty)$ have an empty intersection, except for a possible common starting point.*

(b) *If $d \leq 3$, then, almost surely, the intersection of $(B_1 + f_1)[0, \infty)$ and $(B_2 + f_2)[0, \infty)$ is nontrivial, i.e. contains points other than a possible common starting point.*

Proof of (a). First we will show that $\mathbb{P}(B_2[0, \infty) \text{ intersects } (B_1 + f_1)[0, \infty)) = 0$ and then using the intersection equivalence of B_2 and $B_2 + f_2$ from Theorem 1.5 we will get the result. Recall from the Proof of Theorem 1.5 the definition of the Martin kernel M. Conditioning on $B_2[0, \infty)$ and using the independence of B_1 and B_2, we get from (3.2) that there exist positive constants c_1 and c_2 such that

$$c_1 \mathbb{E}\left(\text{Cap}_M(B_2[0, \infty))\right) \leq \mathbb{P}(B_2[0, \infty) \text{ intersects } (B_1 + f_1)[0, \infty)) \leq c_2 \mathbb{E}\left(\text{Cap}_M(B_2[0, \infty))\right).$$

From [12, Proof of part a, Theorem 9.1] we have that $\mathbb{E}\left(\text{Cap}_M(B_2[0, \infty))\right) = 0$, and hence concluding the proof. $\quad\square$

Proof of (b). If $d = 3$, then, almost surely, $\text{Cap}_M(B_2[0, \infty)) > 0$. We suppose first that $B_1 + f_1$ and $B_2 + f_2$ start from different points, the one from 0 and the other one from x. We then have

$$\mathbb{P}_{0,x}((B_2 + f_2)[0, \infty) \text{ intersects } (B_1 + f_1)[0, \infty)) > c(K, |x|) > 0,$$

where $c(K, |x|)$ is a constant that only depends on the Hölder constant K and on the distance between the starting points. This follows from Proposition 3.1, rotational invariance of Brownian motion and the fact that after a rotation f is still a Hölder$(\frac{1}{2})$ function with the same Hölder constant as f. Also, by scaling invariance of Brownian motion and the fact that $\frac{f(\alpha^2 t)}{\alpha}$ for any $\alpha \neq 0$ is also Hölder$(\frac{1}{2})$ with the same Hölder constant, we get that the probability of intersection is lower bounded by a constant that only depends on the Hölder constant and not on the starting points. Thus we have

$$\mathbb{P}((B_2 + f_2)[0, \infty) \text{ intersects } (B_1 + f_1)[0, \infty)) > c(K) > 0. \tag{4.1}$$

Let $q \leq 1 - c(K) < 1$ be the supremum over the starting points of the probability that $(B_1 + f_1)[0, \infty)$ and $(B_2 + f_2)[0, \infty)$ do not intersect. Then there exists t big enough so that

$$\mathbb{P}(B_2(t_2) + f_2(t_2) \neq B_1(t_1) + f_1(t_1), \text{ for all } 0 < t_1, t_2 \leq t) \leq q + \varepsilon.$$

By the Markov property,

$$q \leq \mathbb{P}(B_2(t_2) + f_2(t_2) \neq B_1(t_1) + f_1(t_1), \forall t_1, t_2 \leq t) \times$$
$$\mathbb{P}(B_2(t_2) + f_2(t_2) \neq B_1(t_1) + f_1(t_1), \forall t_1, t_2 > t) \leq q(q + \varepsilon)$$

and as $\varepsilon > 0$ was arbitrary, we deduce that $q \leq q^2$, and hence $q = 0$. If $B_1 + f_1$ and $B_2 + f_2$ start from the same point, then we write

$$\mathbb{P}(B_2(t_2) + f_2(t_2) \neq B_1(t_1) + f_1(t_1), \text{ for all } t_1, t_2 > 0)$$
$$= \lim_{t \to 0} \mathbb{P}(B_2(t_2) + f_2(t_2) \neq B_1(t_1) + f_1(t_1), \forall t_1, t_2 > t) = 0,$$

which follows from the Markov property of Brownian motion applied to time t.

For dimensions $d < 3$, we project the 3-dimensional motion on the lower dimensional space to obtain nonempty intersection a.s. $\qquad \square$

Proof of Theorem 1.6 (a). We will adapt the proof of [12, Theorem 9.22]. Let

$$X_1(t) = B\left(\frac{1}{2} + t\right) + f\left(\frac{1}{2} + t\right) - B\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) \text{ and}$$
$$X_2(t) = B\left(\frac{1}{2} - t\right) + f\left(\frac{1}{2} - t\right) - B\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right).$$

Then, since X_1 and X_2 are independent, by the independence of the increments of the Brownian motion, we get from Theorem 4.1 that X_1 and X_2 intersect almost surely, thus giving the result. $\qquad \square$

Proof of Theorem 1.6 (b). Let $\alpha \in [0,1]$ be a rational number. We will show that almost surely, there exist no times $0 \le t_1 < \alpha < t_2 \le 1$ with $B(t_1) + f(t_1) = B(t_2) + f(t_2)$. Let X_1 and X_2 be given by

$$X_1(t) = B(\alpha + t) + f(\alpha + t) - B(\alpha) - f(\alpha) \text{ and } X_2(t) = B(\alpha - t) + f(\alpha - t) - B(\alpha) - f(\alpha).$$

Then by the independence of the increments of Brownian motion we get that X_1 and X_2 are independent. By the Hölder assumption on f we get that $f(\alpha + t) - f(\alpha)$ and $f(\alpha - t) - f(\alpha)$ are also Hölder$(\frac{1}{2})$ continuous functions. Hence from Theorem 4.1 we deduce that X_1 and X_2 will not intersect almost surely, thus giving the result. $\qquad\square$

So far we have shown that when the drift f is a Hölder$(\frac{1}{2})$ function, then $B + f$ has no double points in dimension greater than or equal to 4 and has double points in dimension below 4. We will now give an example where adding the drift causes $B + f$ to have double points in dimension greater than 4.

Theorem 4.2. *Let X be a fractional Brownian motion in $d \ge 4$ dimensions with Hurst index $\alpha < \frac{2}{d}$ and let B be an independent standard Brownian motion in d dimensions. Then a.s. $B + X$ has double points.*

Proof. In [10, Theorem 2] it is proven that a fractional Brownian motion in d dimensions with Hurst index $\alpha < \frac{2}{d}$ has double points a.s. Kôno's proof works to show that $B + X$ has double points. The only thing we need to check is that the correlation function r' defined below satisfies the same two inequalities as in [10]. Let a, δ, L be positive numbers and let s, t, u, v be real numbers satisfying:

$$a \le |s - t| \le a + 4\delta, \quad a \le |u - v| \le a + 4\delta$$
$$\min\{|s - u|, |s - v|, |t - u|, |t - v|\} \ge L.$$

We now let

$$r' = \frac{\mathbb{E}((X_s + B_s - X_t - B_t)(X_u + B_u - X_v - B_v))}{(|s - t|^\alpha + \sqrt{|s - t|})(|u - v|^\alpha + \sqrt{|u - v|})}$$
$$= \frac{|t - u|^{2\alpha} + |t - u| + |s - v|^{2\alpha} + |s - v| - |s - u|^{2\alpha} - |s - u| - |t - v|^{2\alpha} - |t - v|}{2(|s - t|^\alpha + \sqrt{|s - t|})(|u - v|^\alpha + \sqrt{|u - v|})}.$$

It is easy to see that by choosing L large enough compared with a and δ we get that $|r'| \le \varepsilon$, for $\varepsilon > 0$. Indeed, define $f(x) = |x - u|^{2\alpha} - |x - v|^{2\alpha}$, for $x < u, v$. Then

$$|r'| \le ca^{-2\alpha}|f(t) - f(s)| = c'a^{-2\alpha}|t - s||u - v||\xi' - \xi|^{2\alpha - 2},$$

where $\xi \in (s, t)$ and $\xi' \in (u, v)$, by applying the mean value theorem twice. Since $2\alpha - 2 < 0$, we get that $|r'| \le \varepsilon$ for L large enough.

We will now explain how we get the second inequality that r' satisfies, namely that

$$1 - r' \ge c(|s - u|^{2\alpha} + |t - v|^{2\alpha})a^{-2\alpha}, \tag{4.2}$$

for a positive constant c, when s, t, u, v are as follows:

$$|s - u| \leq 2\delta, \quad |t - u| \leq 2\delta$$
$$\min\{|s - t|, |s - v|, |t - u|, |u - v|\} \geq a.$$

By translation and scaling it suffices to consider the case when $s = 0, t = 1$ and $v = 1 + \gamma$ and wlog we assume that $u < \gamma$. Thus it suffices to show that

$$4((1 + \gamma - u)^\alpha + (1 + \gamma - u)^{\frac{1}{2}}) - (1 - u)^{2\alpha} - 2 + 2u - (1 + \gamma)^{2\alpha} + u^{2\alpha} + \gamma^{2\alpha} \geq c\gamma^{2\alpha},$$

for a positive constant c. Using Taylor expansion to first order terms and using the fact that $\alpha < \frac{2}{d}$, i.e. $2\alpha < 1$, and that we can make γ as small as we like since it is smaller than δ, we get inequality (4.2). $\qquad \square$

5 Hausdorff dimension of the image and graph

We will now state a classical result that will be used in the proof of Theorem 1.8. We include its proof here for the sake of completeness.

Lemma 5.1. *Let (A, d) be a compact metric space and $f : A \to \mathbb{R}$ a continuous function. If ν is a probability measure on $K = f(A)$, then there exists a probability measure μ on A such that $\nu = \mu \circ f^{-1}$.*

Proof. Define a linear functional Ψ on the closed subspace $\Upsilon = \{\varphi \circ f | \varphi \in \mathcal{C}(K)\}$ of $\mathcal{C}(A)$ by $\Psi(\varphi \circ f) = \int_K \varphi \, d\nu$. Clearly, $\|\Psi\| = 1$. By the Hahn-Banach theorem, Ψ can be extended to a linear functional $\tilde{\Psi}$ on $\mathcal{C}(A)$ of norm 1. Since $\tilde{\Psi}(1) = 1$, by the Riesz-representation theorem, there exists a probability measure μ on A such that $\tilde{\Psi}(g) = \int_A g \, d\mu$, for all $g \in \mathcal{C}(A)$. $\qquad \square$

Proof of Theorem 1.8. We will first show that $\dim(B + f)(A) \geq (2 \dim A) \wedge d$ a.s. Let $\alpha < \dim A \wedge \frac{d}{2}$. Then by Theorem 2.7, there exists a probability measure μ on A with finite α-energy, i.e. $\mathcal{E}_\alpha(\mu) < \infty$. We now define a random measure $\tilde{\mu}$ on $(B + f)(A)$ given by $\tilde{\mu}(D) = \mu((B + f)^{-1}(D))$, for any $D \subset (B + f)(A)$. Thus if we show that

$$\mathbb{E}(\mathcal{E}_{2\alpha}(\tilde{\mu})) = \mathbb{E} \int \int \frac{d\tilde{\mu}(x) \, d\tilde{\mu}(y)}{|x - y|^{2\alpha}} = \mathbb{E} \int_0^1 \int_0^1 \frac{d\mu(s) \, d\mu(t)}{|B_t + f(t) - B_s - f(s)|^{2\alpha}} < \infty,$$

then using Theorem 2.7 we will conclude that $\dim(B + f)(A) \geq 2\alpha$ a.s. and hence by letting $\alpha \uparrow \dim A \wedge \frac{d}{2}$ we will get the result.

Applying Fubini we get that

$$\mathbb{E}(\mathcal{E}_{2\alpha}(\tilde{\mu})) = \int_0^1 \int_0^1 \mathbb{E} \left(\frac{1}{|B_t + f(t) - B_s - f(s)|^{2\alpha}} \right) d\mu(s) \, d\mu(t),$$

so we need to estimate the expectation appearing in the last integral.

Take $t > s$, and write $\beta = \sqrt{t-s}$ and $u = f(s) - f(t)$. Then we want to evaluate

$$\mathbb{E}\frac{1}{|\beta B_1 - u|^{2\alpha}} = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{e^{-\frac{|x|^2}{2}}}{|\beta x - u|^{2\alpha}}\,dx = \frac{1}{|\beta|^{2\alpha}} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{e^{-\frac{|x|^2}{2}}}{|x - \frac{u}{\beta}|^{2\alpha}}\,dx.$$

Define $g(x) = e^{-\frac{|x|^2}{2}}$ and $h(x) = \frac{1}{|x|^{2\alpha}}$. They are both decreasing functions of $|x|$ and hence using that for all $w \in \mathbb{R}^d$ we have that

$$\int_{\mathbb{R}^d} (g(x) - g(x - w))(h(x) - h(x - w))\,dx \geq 0, \tag{5.1}$$

we get that

$$\frac{1}{|\beta|^{2\alpha}} \int \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{e^{-\frac{|x|^2}{2}}}{|x - \frac{u}{\beta}|^{2\alpha}}\,dx \leq \int \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{e^{-\frac{|x|^2}{2}}}{|\beta x|^{2\alpha}}\,dx. \tag{5.2}$$

Hence we obtain that

$$\mathbb{E}\frac{1}{|B_t + f(t) - B_s - f(s)|^{2\alpha}} \leq \mathbb{E}\frac{1}{|B_t - B_s|^{2\alpha}}$$

and thus

$$\mathbb{E}(\mathcal{E}_{2\alpha}(\tilde{\mu})) \leq \mathbb{E} \int_0^1 \int_0^1 \frac{d\mu(s)\,d\mu(t)}{|B_t - B_s|^{2\alpha}}.$$

The rest of the proof follows in the same way as the proof of McKean's theorem (see for instance [12, Theorem 4.33]).

We will now show that $\dim(B + f)(A) \geq \dim f(A)$ a.s. Let $\alpha < \dim f(A) \leq d$. Then again by Theorem 2.7, there exists a probability measure ν on $K = f(A)$ with finite α-energy, that is $\mathcal{E}_\alpha(\nu) < \infty$. By Lemma 5.1, we can find a probability measure $\tilde{\nu}$ on A such that $\nu = \tilde{\nu} \circ f^{-1}$. Then

$$\int_K \int_K \frac{d\nu(x)\,d\nu(y)}{|x - y|^\alpha} = \int_0^1 \int_0^1 \frac{d\tilde{\nu}(s)\,d\tilde{\nu}(t)}{|f(s) - f(t)|^\alpha} < \infty. \tag{5.3}$$

We finally define a random measure μ' on $K_f = (B + f)(A)$ by $\mu'(D) = \tilde{\nu}((B + f)^{-1}(D))$, for any $D \subset K_f$. We want to show that

$$\mathbb{E}(\mathcal{E}_\alpha(\mu')) < \infty,$$

and hence using the energy method again we will deduce that $\dim(B+f)(A) \geq \alpha$ a.s. Finally, just as before, letting $\alpha \uparrow \dim f(A)$ we will obtain that almost surely $\dim(B + f)(A) \geq \dim f(A)$.

We now have that

$$\mathbb{E}(\mathcal{E}_\alpha(\mu')) = \mathbb{E} \int_{K_f} \int_{K_f} \frac{d\mu'(x)\,d\mu'(y)}{|x - y|^\alpha} = \mathbb{E} \int_0^1 \int_0^1 \frac{d\tilde{\nu}(s)\,d\tilde{\nu}(t)}{|B_s + f(s) - B_t - f(t)|^\alpha}.$$

Take $s > t$ and set $\beta = \sqrt{s-t}$ and $w = f(s) - f(t)$.

$$\mathbb{E}\frac{1}{|B_s + f(s) - B_t - f(t)|^\alpha} = \mathbb{E}\frac{1}{|\beta B_1 + w|^\alpha} = \mathbb{E}\frac{1}{\beta^\alpha|B_1 + \frac{w}{\beta}|^\alpha} = \frac{1}{\beta^\alpha} \int \frac{1}{(2\pi)^{\frac{d}{2}}|x + \frac{w}{\beta}|^\alpha} e^{-\frac{|x|^2}{2}}\,dx.$$

Let $u = \frac{w}{\beta}$. Then

$$\int_{\mathbb{R}^d} \frac{1}{|x+u|^\alpha} e^{-\frac{|x|^2}{2}} \, dx = \int_{|x+u| \geq \frac{|u|}{2}} \frac{1}{|x+u|^\alpha} e^{-\frac{|x|^2}{2}} \, dx + \int_{|x+u| < \frac{|u|}{2}} \frac{1}{|x+u|^\alpha} e^{-\frac{|x|^2}{2}} \, dx$$

$$\leq \frac{c_1}{|u|^\alpha} + e^{\frac{-|u|^2}{4}} \int_{|x| < |u|} \frac{1}{|x|^\alpha} \, dx = \frac{c_1}{|u|^\alpha} + c_2 e^{\frac{-|u|^2}{4}} \int_0^{|u|} \frac{r^{d-1}}{r^\alpha} \, dr \leq \frac{c_3}{|u|^\alpha},$$

since $d > \alpha$. Hence,

$$\mathbb{E} \frac{1}{|B_s + f(s) - B_t - f(t)|^\alpha} \leq \frac{C}{|f(t) - f(s)|^\alpha}$$

and thus using (5.3) we deduce that

$$\mathbb{E}(\mathcal{E}_\alpha(\mu')) < \infty.$$

$\square$

Remark 5.2. We note that when f is a γ-Hölder continuous function, then $\dim f(A) \leq \frac{1}{\gamma} \dim A$ and similarly for Minkowski dimension. Hence using the a.s. α-Hölder continuity property of B for any $\alpha < \frac{1}{2}$, we deduce that $\dim(B+f)(A) \leq \max\{2, \frac{1}{\gamma}\} \dim A$. We also note that for any function f we have that almost surely

$$\dim(B+f)(A) \leq \dim_M B(A) + \dim f(A) \leq 2 \dim_M A + \dim f(A), \tag{5.4}$$

where $\dim_M$ denotes the Minkowski dimension. The first inequality follows from [11, Theorem 8.10] and the second from the Hölder property of Brownian motion.

Remark 5.3. We note that the lower bound for the Hausdorff dimension of the image of $B+f$ given in Theorem 1.8 is sharp in some cases. For example let X be an independent fractional Brownian motion in $d \geq 2$ dimensions of Hurst index α. Then it is β-Hölder continuous for any $\beta < \alpha$ and it is known that almost surely $\dim(\text{Im}X) = \frac{1}{\alpha} \wedge d$ ([8, Chapter 18]). Therefore $\dim(\text{Im}(B+X)) = \max\{2, \frac{1}{\alpha} \wedge d\}$ a.s.

We now give an example where the lower bound from Theorem 1.8 is not sharp and where the upper bound given in (5.4) is almost sharp.

Example 5.4. Let B be a standard Brownian motion in 3 dimensions and let $f(t) = (f_1(t), 0, 0)$, where f_1 is a fractional Brownian motion independent of B of Hurst index α. Then $\dim f[0, 1] = 1$ a.s. For α small, we have that almost surely $\dim(B+f)[0, 1] = 3 - 2\alpha$, which is a special case of [4, Theorem 1].

So far we have obtained bounds for the Hausdorff dimension of the image of $B + f$. Similar results to the ones proved for the image hold also for the graph and the proofs use the same methods as before. We will only point out the parts where they differ.

Proof of Theorem 1.9 (a). To show that $\dim G(B+f) \geq \frac{3}{2}$, we will adapt the proof of [12, Theorem 4.29 a]. Let $\alpha < \frac{3}{2}$. Define a random measure μ on the graph of $B+f$ by

$$\mu(A) = \mathcal{L}(\{0 \leq t \leq 1 : (t, B_t + f(t)) \in A\}).$$

We want to show that this measure has finite α-energy, i.e. that

$$\mathbb{E}\int\int\frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} = \mathbb{E}\int_0^1\int_0^1\frac{ds\,dt}{(|t-s|^2+|B_t+f(t)-B_s-f(s)|^2)^{\frac{\alpha}{2}}} < \infty.$$

Using again (5.1) as in the proof of Theorem 1.8 we get that

$$\mathbb{E}\frac{1}{(|t-s|^2+|B_t+f(t)-B_s-f(s)|^2)^{\frac{\alpha}{2}}} \le \mathbb{E}\frac{1}{(|t-s|^2+|B_t-B_s|^2)^{\frac{\alpha}{2}}}.$$

Now the rest of the proof follows just as in [12, Theorem 4.29 a].

It remains to show that $\dim G(B+f) \ge \dim G(f)$. Let $\alpha < \dim G(f)$. Then there exists a measure ν on $G(f)$ such that $\mathcal{E}_\alpha(\nu) < \infty$. Then there exists a measure μ on $[0,1]$ such that

$$\nu(A) = \mu(\{t : (t,f(t)) \in A\}).$$

Next we define a random measure μ' on $G(B+f)$ by $\mu'(A) = \mu(\{t : (t,B_t+f(t)) \in A\})$. We will show that
$\mathbb{E}(\mathcal{E}_\alpha(\mu')) < \infty$, i.e.

$$\mathbb{E}\int_0^1\int_0^1\frac{d\mu(s)\,d\mu(t)}{(|t-s|^2+|B_t+f(t)-B_s-f(s)|^2)^{\frac{\alpha}{2}}} < \infty.$$

Using the same arguments as in the proof of Theorem 1.8 we get that

$$\mathbb{E}\frac{1}{(|t-s|^2+|B_t+f(t)-B_s-f(s)|^2)^{\frac{\alpha}{2}}} \le \frac{C}{(|t-s|^2+|f(t)-f(s)|^2)^{\frac{\alpha}{2}}}$$

and thus the result follows since $\mathbb{E}(\mathcal{E}_\alpha(\nu)) < \infty$. $\qquad\square$

Proof of Theorem 1.9 (b). We have that almost surely $\dim G(B+f) \ge 2$ from Theorem 1.8, since the dimension of the graph is always bigger than the dimension of the image. We only need to show that it is greater than the dimension of the graph of f. To prove that we use the energy method just as in the proof for the case $d=1$. $\qquad\square$

Question

Can the zero-one law (Theorem 1.1) be extended to Gaussian processes X with dependent increments, e.g. fractional Brownian motion or the Gaussian Free Field in a domain $U \subset \mathbb{R}^d$? For example, does $\{\mathcal{L}(X+f)(A) > 0\}$ satisfy a 0-1 law?

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