QLE

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August 1, 2013

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- **SLE:** Schramm Loewner evolution. Random fractal curve related to LQG and GFF, and to various discrete random paths. Defined for real $\kappa \geq 0$.

A couple of big questions

- LQG is a conformal structure with an area measure, and TBM is a metric with an area measure. Is there a natural way to put a conformal structure on TBM, or a metric space structure on LQG, that would give a coupling between these two objects?

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- Can one say anything at all about any kind of scaling limit of any kind of DLA? Note: throughout this talk we use DLA to refer to *external DLA*. The so-called *internal DLA* is a process that grows spherically with very small (log order) fluctuations, smaller than those of KPZ growth processes. There has been more mathematical progress on internal DLA. (I was part of recent IDLA paper series with Levine and Jerison.)

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- For fun, I browsed through the first 500 (of $11,700$) articles on "diffusion limited aggregation" listed at scholar.google.com. I found only four in math journals. Three about internal DLA (works by Lawler; Lawler, Bramson, Griffeath; Blachere, Brofferio) and one about DLA on a tree (Barlow, Pemantle, Perkins).

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- More math papers listed at mathscinet, including Kesten's $n^{2/3}$ upper bound on diameter after n steps. In his ICM paper, Schramm called this "essentially the only theorem concerning two-dimensional DLA".

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- **Question:** Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rate is non-linear (power-law) function of harmonic exposure?

Complicating life further

- Can we make sense of α-DLA on a γ-LQG? There is a way to tile an LQG surface with diadic squares of "about the same size" (see next slide) so we could to DLA on this set of squares and try to take a fine mesh limit.

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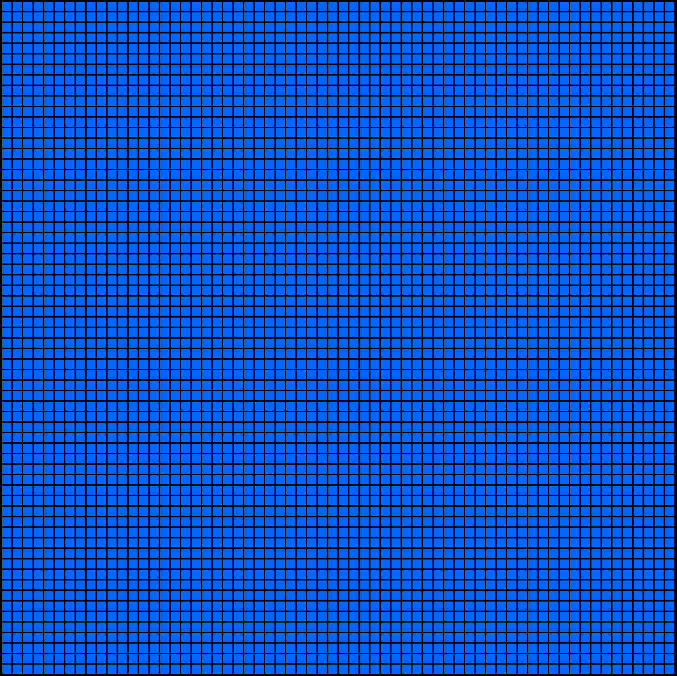
- We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution:** QLE(γ^2, α).
- But first let's look at some simulations/animations.

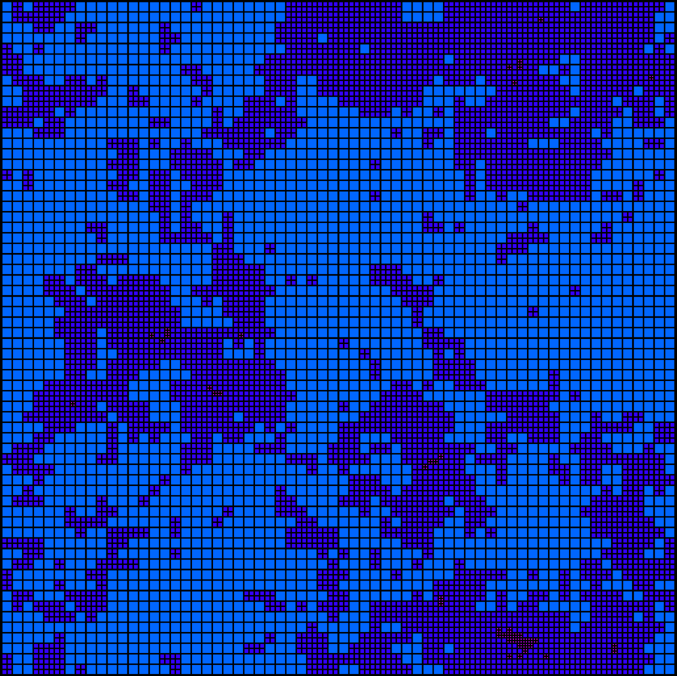
Constructing the random metric

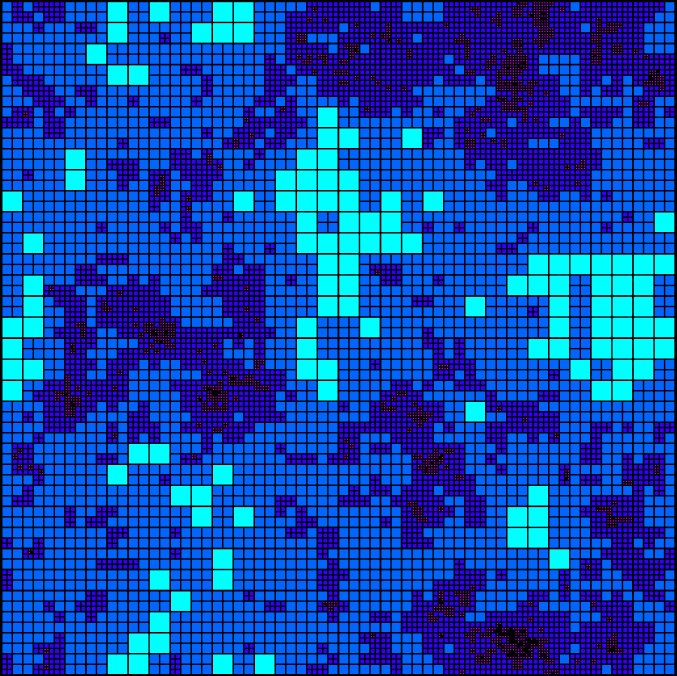
Let $h_\epsilon(z)$ denote the mean value of h on the circle of radius ϵ centered at z. This is almost surely a locally Hölder continuous function of (ϵ, z) on $(0,\infty) \times D$. For each fixed ϵ, consider the surface $\mathcal{M}_\epsilon$ parameterized by D with metric $e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$.

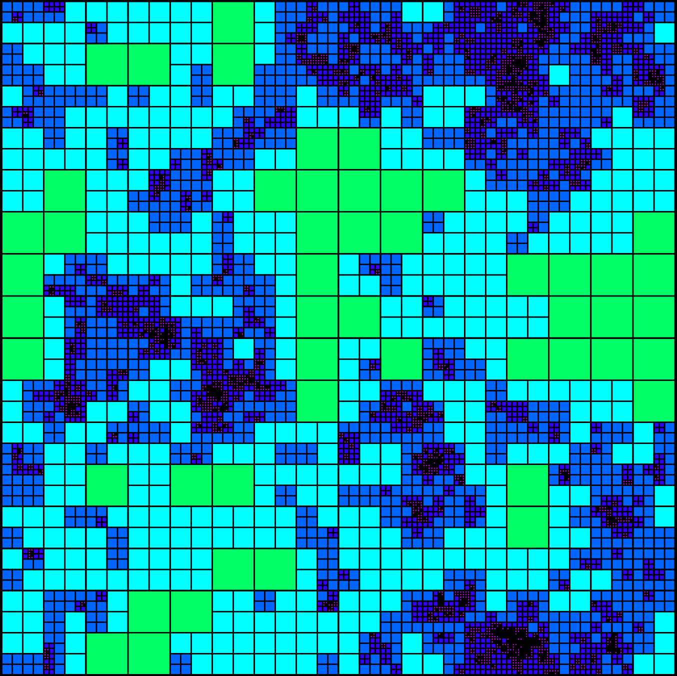
We define $\mathcal{M} = \lim_{\epsilon \to 0} \mathcal{M}_\epsilon$, but what does that mean?

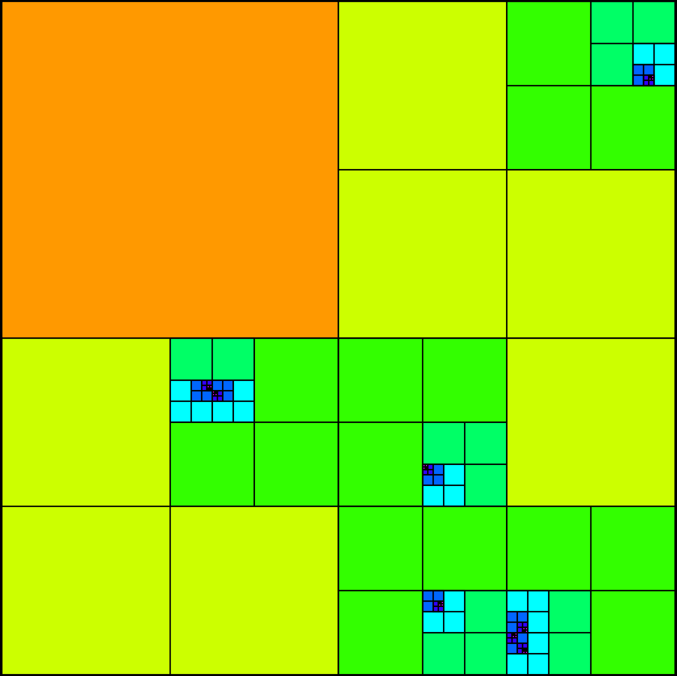
PROPOSITION: Fix $\gamma \in [0,2)$ and define h, D, and μ_ϵ as above. Then it is almost surely the case that as $\epsilon \to 0$ along powers of two, the measures $\mu_\epsilon := \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz$ converge weakly to a non-trivial limiting measure, which we denote by $\mu = \mu_h = e^{\gamma h(z)} dz$.











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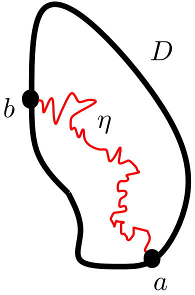
- Let's recall how SLE was defined.

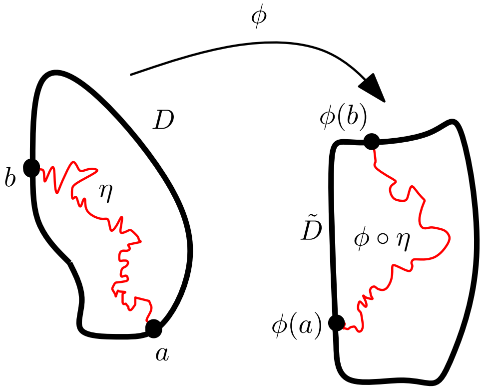
A canonical measure on non-self-crossing paths

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random non-self-crossing path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is. Would like to argue that SLE is in some sense the "canonical" random non-self-crossing path. What symmetries characterize SLE?

Conformal Markov property of SLE

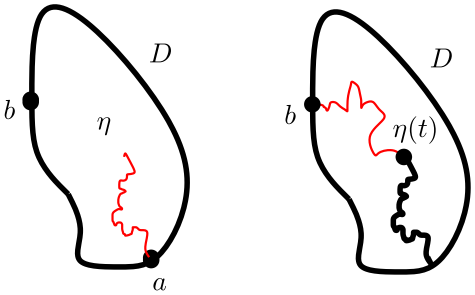


If ϕ conformally maps D to $\tilde{D}$ and η is an SLE_κ from a to b in D, then $\phi \circ \eta$ is an SLE_κ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

Markov Property

Given η up to a stopping time t...

law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



Chordal Schramm-Loewner evolution (SLE)

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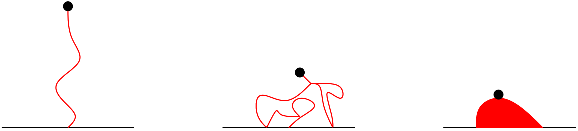
- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.
- **Explicit construction:** An SLE path γ from 0 to ∞ in the complex upper half plane $\mathbb{H}$ can be defined in an interesting way: given path γ one can construct conformal maps $g_t : \mathbb{H} \setminus \gamma([0, t]) \to \mathbb{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z\to\infty} g_t(z) - z = 0$). In SLE$_\kappa$, one defines g_t via an ODE (which makes sense for each fixed z):

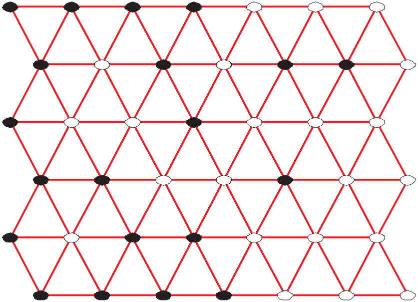
$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

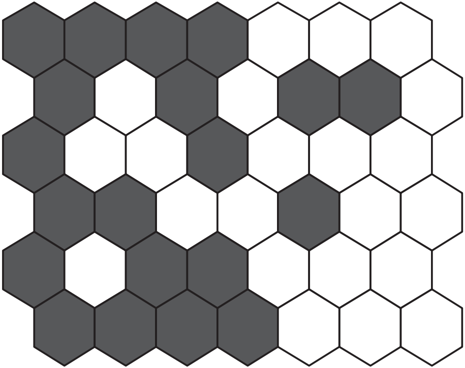
where $W_t = \sqrt{\kappa} B_t =_{LAW} B_{\kappa t}$ and B_t is ordinary Brownian motion.

SLE phases [Rohde, Schramm]



$\kappa \leq 4$ $\quad$ $\kappa \in (4, 8)$ $\quad$ $\kappa \geq 8$





Radial Schramm-Loewner evolution (SLE)

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- **Radial measure-driven Loewner evolution:** $\partial g_t(z) = \int g_t(z)\frac{x+g_t(z)}{x-g_t(z)} dm_t(x)$ where, for each g, m_t is a measure on the complex unit circle.

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- Should we take subsequential limit of α-DLA on γ-RPM (or some isotropic/Markovian variant) and *define* that to be QLE(γ^2, α)?
- Maybe, but aside from uniqueness issue, this wouldn't tell us what kind of measure-valued driving function we have, whether limit process is "simple" in sense that it doesn't absorb positive area "bubbles' in zero time, whether all space is ultimately absorbed, what the quantum dimension of the "trace" should be, what stationary law of the random measure is, whether the evolving random measure is a Markovian process on the space of measures (as one would expect).

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Measure-driving Loewner evolution

- Space of measure-driven Loewner evolutions (unlike space of point-driven Loewner evolutions) is compact.
- Related to fact that space of Lipschitz functions is compact, and any Lipschitz function is integral of its a.e. defined derivative.
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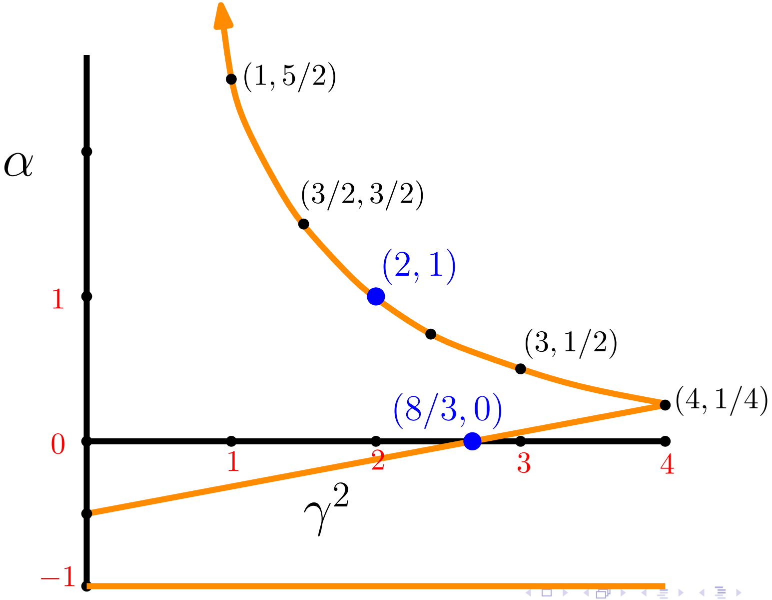
- Yes, at least for (γ^2, α) pairs. Surprising connection to SLE.

Three important QLE families



FPP vs. percolation interface, DLA vs. LERW

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- Scaling limits should be $\mathrm{QLE}(8/3, 0)$ and $\mathrm{QLE}(2, 1)$.

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- We understand very well how to draw an SLE coupled with a random surface for a fixed amount of quantum time, and then resample the seed origin from the appropriate geometric combination of μ and ν (harmonic and quantum measures).
- These results are related to the radial form of the so-called "quantum zipper", which comes from drawing whole plane SLE, targeted at an interior point, on top of an LQG.

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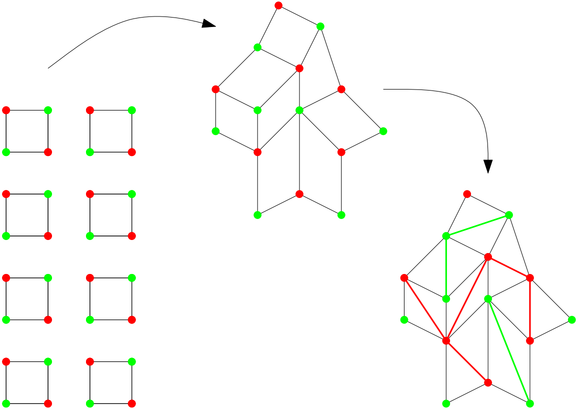
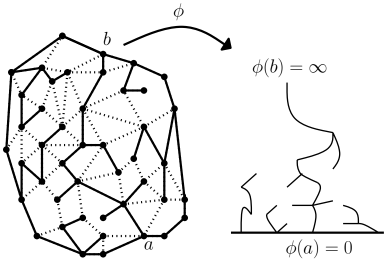
- **Continuum approach**: As described above, use conformal maps to reduce to a problem of constructing a random real-valued function on a planar domain or a sphere. Using the Gaussian free field for the random function yields (critical) Liouville quantum gravity.

Discrete construction: gluing squares



Discrete uniformizing maps



Planar map with one-chord-wired spanning tree (solid edges), plus image under conformal map to $\mathbb{H}$ (sketch).

How about the continuum construction? Defining Liouville quantum gravity? Takes some thought because h is distribution not function.

Changing coordinates

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- Similarly, the boundary length ν_h is almost surely the image under ψ of the measure $\nu_{\tilde{h}}$.

Defining *quantum surfaces*

- **DEFINITION:** A **quantum surface** is an equivalence class of pairs (D, h) under the equivalence transformations
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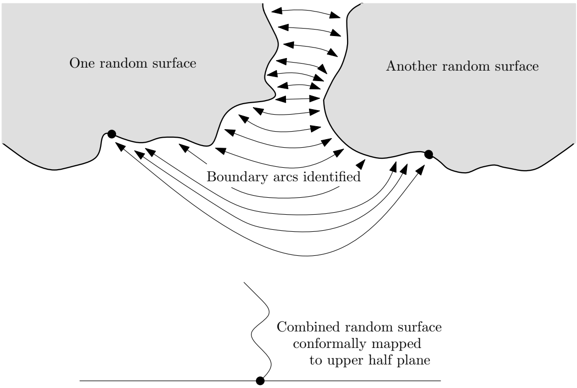
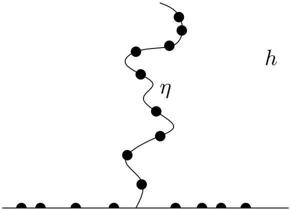
- Area, boundary length, and conformal structure are well defined for such surfaces.

Glue two random surfaces: interface is random path

Theorem [S.]: If you glue two appropriate independent random quantum surfaces along their boundaries (in a length preserving way) and conformally map the new surface you get back to the half plane, then the image of the interfaces becomes an SLE.



Stationarity and matching quantum lengths



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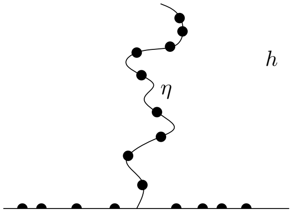

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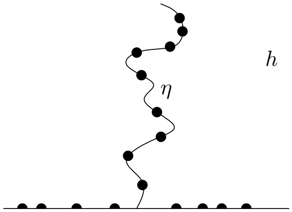

- Sketch of interface path η with marks spaced at intervals of equal ν_h length.
- The random pair (h, η) is stationary with respect to zipping up or down by a unit of (capacity) time.
- In this pair, h and η are (surprisingly) actually independent of each other.

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- The planar map versions of QLE$(8/3, 0)$ and QLE$(2, 1)$ described earlier should correspond to $\kappa = 6$ and $\kappa = 2$.
- It seems that all of the mass is at the tip when $\kappa \leq 1$, suggesting that this procedure just produces an an ordinary path in that case. Kind of makes sense.

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- This construction also produces geodesics.

Results (in various stages...)

- **Construction of QLE as local set of GFF:** For (α, γ^2) pairs along the curves shown, one can explicitly write down the stationary law of the Loewner driving measure on the circle boundary and show that this law is exactly preserved by both ϵ-time-jump approximations and their limits.

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- **An at-long-last TBM/LQG link:** We think we can show that the $(8/3, 0)$ case produces a metric with the same law as the TBM, and that TBM structure is a.s. determined by the LQG.

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