

**1.12 Appendix: asymptotics for  $n!$** 

Our analysis of recurrence and transience for random walks in Section 1.6 rested heavily on the use of the asymptotic relation

$$n! \sim A\sqrt{n}(n/e)^n \quad \text{as } n \rightarrow \infty$$

for some  $A \in [1, \infty)$ . Here is a derivation.

We make use of the power series expansions for  $|t| < 1$

$$\begin{aligned} \log(1+t) &= t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \dots \\ \log(1-t) &= -t - \frac{1}{2}t^2 - \frac{1}{3}t^3 - \dots \end{aligned}$$

By subtraction we obtain

$$\frac{1}{2} \log \left( \frac{1+t}{1-t} \right) = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \dots$$

Set  $A_n = n!/(n^{n+1/2}e^{-n})$  and  $a_n = \log A_n$ . Then, by a straightforward calculation

$$a_n - a_{n+1} = (2n+1) \frac{1}{2} \log \left( \frac{1+(2n+1)^{-1}}{1-(2n+1)^{-1}} \right) - 1.$$

By the series expansion written above we have

$$\begin{aligned} a_n - a_{n+1} &= (2n+1) \left\{ \frac{1}{(2n+1)} + \frac{1}{3} \frac{1}{(2n+1)^3} + \frac{1}{5} \frac{1}{(2n+1)^5} + \dots \right\} - 1 \\ &= \frac{1}{3} \frac{1}{(2n+1)^2} + \frac{1}{5} \frac{1}{(2n+1)^4} + \dots \\ &\leq \frac{1}{3} \left\{ \frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots \right\} \\ &= \frac{1}{3} \frac{1}{(2n+1)^2 - 1} = \frac{1}{12n} - \frac{1}{12(n+1)}. \end{aligned}$$

It follows that  $a_n$  decreases and  $a_n - 1/(12n)$  increases as  $n \rightarrow \infty$ . Hence  $a_n \rightarrow a$  for some  $a \in [0, \infty)$  and hence  $A_n \rightarrow A$ , as  $n \rightarrow \infty$ , where  $A = e^a$ .