1.2 Class structure

It is sometimes possible to break a Markov chain into smaller pieces, each of which is relatively easy to understand, and which together give an understanding of the whole. This is done by identifying the communicating classes of the chain.

We say that i *leads to* j and write $i \to j$ if

$$P_i(X_n = j \text{ for some } n \geq 0) > 0.$$

We say i *communicates with* j and write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

Theorem 1.2.1. *For distinct states i and j the following are equivalent:*

(i) $i \to j$;

(ii) $p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n} > 0$ *for some states* $i_0, i_1, \dots, i_n$ *with* $i_0 = i$ *and* $i_n = j$;

(iii) $p_{ij}^{(n)} > 0$ *for some* $n \geq 0$.

Proof. Observe that

$$p_{ij}^{(n)} \leq P_i(X_n = j \text{ for some } n \geq 0) \leq \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

which proves the equivalence of (i) and (iii). Also

$$p_{ij}^{(n)} = \sum_{i_1, \dots, i_{n-1}} p_{i i_1} p_{i_1 i_2} \dots p_{i_{n-1} j}$$

so that (ii) and (iii) are equivalent. □

It is clear from (ii) that $i \to j$ and $j \to k$ imply $i \to k$. Also $i \to i$ for any state i. So $\leftrightarrow$ satisfies the conditions for an equivalence relation on I, and thus partitions I into *communicating classes*. We say that a class C is *closed* if

$$i \in C, i \to j \quad \text{imply } j \in C.$$

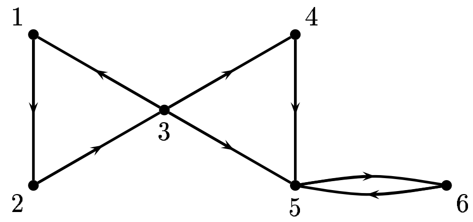
Thus a closed class is one from which there is no escape. A state i is *absorbing* if $\{i\}$ is a closed class. The smaller pieces referred to above are these communicating classes. A chain or transition matrix P where I is a single class is called *irreducible*.

As the following example makes clear, when one can draw the diagram, the class structure of a chain is very easy to find.

Example 1.2.2

Find the communicating classes associated to the stochastic matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$



The solution is obvious from the diagram
the classes being $\{1, 2, 3\}$, $\{4\}$ and $\{5, 6\}$, with only $\{5, 6\}$ being closed.

Exercises

1.2.1 Identify the communicating classes of the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Which classes are closed?

1.2.2 Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.