NEW FRONTIERS IN RANDOM GEOMETRY (RaG)
EP/103372X/1
REPORT 1/7/12 – 30/6/13

NATHANAËL BERESTYCKI, GEOFFREY GRIMMETT, AND JAMES NORRIS

1. Management Process

The Management Committee (MC) comprises the three investigators and the three members of the external Advisory Board (AB), namely Yuval Peres, Stanislav Smirnov, and Wendelin Werner. The local managers have met weekly during term, and more formally about every two months. The advice of the AB has been sought on a variety of matters including the hiring process. One member of the AB (Werner) has spent a considerable period in Cambridge during the period of this report.

A meeting of the AB took place on 24 August 2012. Yuval Peres and Stanislav Smirnov participated by telephone (minutes circulated earlier).

2. Personnel

One postdoctoral research fellow was appointed following the advertisement of December 2012, and will take up post on 1 September 2013.

- Laure Dumaz[1], MMath (Ecole Normale Supérieure, 2010), PhD (ENS, Technical University of Budapest, 2012), from 1 September 2013 to 31 August 2015.

At the time of the award of the grant, Cambridge University agreed to the creation of a new tenured post in probability. This post was filled during the period of this report by the appointment of Paul Bourgade.

Hugo Dumini-Copin and Jason Miller have joined the team as nominated researchers.

3. Research Programme (selected)

3.1. **Hastings-Levitov conformal aggregation and related growth models.** Johansson Viklund, Norris, Sola, and Turner are studying a regularized version of the Hastings–Levitov growth model. A sequence of growing clusters is obtain by composing random conformal maps; the nth map adds a single particle of size d_n to the unit disk. The numbers d_n are obtained

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http://www.statslab.cam.ac.uk/~grg/rag.html.

[1] http://www.math.ens.fr/~dumaz/

by scaling a basic parameter d by the αth power of the derivative of the composed map Φ_{n-1}, evaluated at distance $\sigma > 0$ away from the boundary.

We have established that the clusters converge to growing disks as $d \to 0$ and $n = d^{-2}$; the regularization parameter is allowed to tend to 0 with d, provided it does so sufficiently slowly. Over longer time-scales, one can study the internal structure of the clusters, and by letting $\alpha d^2 \to a \in [0, \infty]$ with d, one extracts a family of limiting objects described in terms of variants of the Brownian web. The proofs involve coupling $\{\Phi_n\}$ with an auxiliary sequence of conformal maps that are amenable to the analysis previously carried out for $\alpha = 0$ by Norris and Turner, and controlling an associated recursive error. The estimates used in these proofs break down when $\sigma^2 \log d^{-1} < 1$, and it seems a more refined analysis is required to obtain results for smaller values of σ.

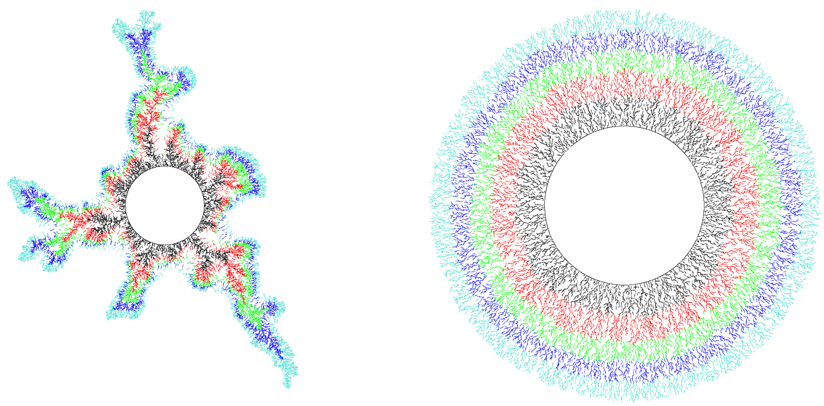

FIGURE 3.1. HL$(2, \sigma)$ clusters with $\sigma = d$ and $\sigma \gg d$, grown with $d = 0.02$ and $n \asymp d^{-1}$. Different colours indicate epochs of 5,000 arrivals.

Systematic simulation of the sequences $\{d_n\}$, and of the resulting clusters and circle flows, has been crucial in formulating precise statements and testing proof strategies. From these simulations it seems clear that purely random scaling limits will appear when σ is comparable to d. It is a long-term objective to extract such limits.

3.2. **Cyclicity in Hilbert spaces of analytic functions.** For many reasonable Hilbert spaces X of analytic functions in the unit disk, multiplication by the coordinate function induces an operator $S\colon X \to X$ via $Sf(z) = zf(z)$, $f \in X$. A classical topic in complex and harmonic analysis is to classify the invariant subspaces of X with respect to this operator, and a fundamental question is for which $f \in X$ we have that f is cyclic in X with respect to S. This last property can be rephrased in terms of the existence of a sequence $\{p_n\}$ of polynomials having $\|p_n f - 1\|_X \to 0$.

Bénéteau, Condori, Liaw, Seco, and Sola were able to identify optimal sequences of polynomials for this problem for certain classes of functions, and to obtain sharp estimates on rates of convergence of norms.

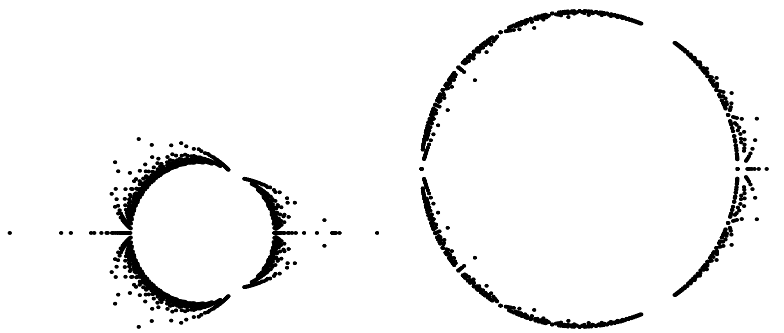

FIGURE 3.2. Zero sets of the first 50 optimal polynomials and Taylor polynomials, respectively, in the cyclicity problem in the Dirichlet space for a function having two zeros on the unit circle.

3.3. **Universality of isoradial dimers.** An isoradial graph is a graph which can be embedded into the plane such that each face is inscribable into a circle of common radius. For a given simply-connected domain, Li has used isoradial graphs with 'special' boundary conditions to approximate the simply-connected domain, as the common radius goes to 0, to prove that the distribution of height function of perfect matchings is conformally invariant and converges to a Gaussian free field. This answers a question by Chelkak and Smirnov (*Invent. Math.* 2012).

3.4. **Critical percolation probability.** Grimmett and Li are investigating an old question of Benjamini and Schramm, namely whether the critical percolation probabilities are strictly different for a graph G and the quotient graph G/Γ, for some non-trivial group Γ of automorphisms. They have made progress on this problem, which has been curiously elusive. The special case of Cayley graphs is informative.

3.5. **Self-avoiding walks.** Grimmett and Li have largely completed their project on understanding the behaviour of the connective constant $\mu = \mu(G)$ as the underlying graph G varies. In their last two papers on this topic, they have proved a lower bound for μ for regular, transitive graphs, and have found necessary and sufficient conditions for μ to be strictly different for a quotient graph (see Project 3.4 above). Certain largely algebraic questions remain open.

This work has led to an ongoing study by Grimmett, Holroyd, and Peres of the growth rates of so-called *extendable* self-avoiding walks.

FIGURE 3.3. A simulation of a uniform random planar triangulation

3.6. **Brownian motion in Liouville Quantum Gravity.** Liouville Quantum Gravity is a model for random planar geometry, and can be informally described as being the Riemann surface corresponding to the tensor $\rho(z) = \exp(\gamma h(z))$, where $\gamma \geq 0$ and $h(z)$ is a Gaussian Free Field. It is widely believed that the scaling limit of discrete uniform planar random maps are given thus with $\gamma = \sqrt{8/3}$.

The exact way to define a metric in the plane remains unclear as the Gaussian Free Field is too rough for its exponential to be well-defined. Berestycki has augmented an earlier result of Duplantier and Sheffield by constructing the natural Brownian motion associated with the metric for certain parameter values. (A similar result has been proved independently by Garban, Rhodes, and Vargas.) One of the outcomes is that, almost surely at a given time, the Brownian motion is 'stuck' in an atypical thick point of the Gaussian Free Field.

Many open questions on this topic are currently being investigated, some of which jointly with Garban, Rhodes, and Vargas, in particular the supercritical case $\gamma > 2$ and corresponding duality properties (which should mirror the duality properties between SLE_κ and $\mathrm{SLE}_{16/\kappa}$).

3.7. **Interchange process and representation theory.** Berestycki and Kozma have identified a useful new approach to the interchange process, based on representation theory. This has yielded an elegant proof of a result of Schramm on the topic, together with new exact and simple formulae.

3.8. **Wulff crystal random walk.** Berestycki and Yadin have considered a Gibbs distribution on random paths in $\mathbb{Z}^2$ in which the weight of path ω is proportional to $\exp(-\beta|\partial R_t|)$, where $\beta > 0$ and ∂R_t is the set of boundary vertices of ω. This gives a natural random walk construction of the Wulff crystal. The main result is that, for β small, the diameter of the path is

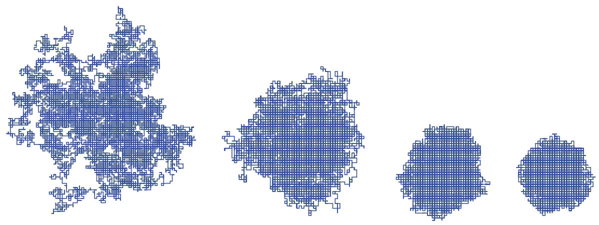

FIGURE 3.4. Wulff crystal random walk simulations for various values of β.

of order $t^{1/3}$, up to logarithmic corrections, thus confirming predictions of physicists.

3.9. **Schramm–Smirnov topology and coalescing flows.** Berestycki, Garban, and Sen are completing work on scaling limits of coalescing flows, making use of prior work of Schramm and Smirnov on scaling limits of critical percolation. They obtain, for example, convergence of a discrete flow of coalescing random walks to Arratia's flow under optimal moment assumptions on the random walk, and this extends to environments with more complicated geometry, such as the Sierpinski gasket. A second part on applications of these techniques to the 'black noise property' is in preparation.

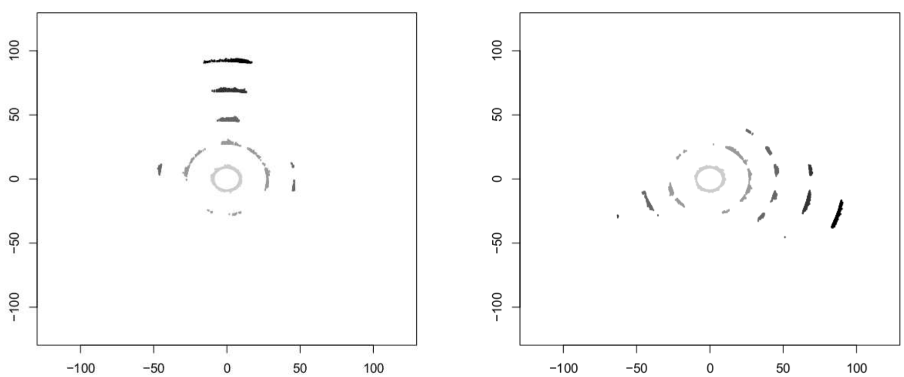

FIGURE 3.5. Two realisations of the particle system with $N = 1000$, $d = 2$, $s(x, y) = x^2 + y^2$. The particles are plotted after 20, 60, 100, 150 and 200 generations with decreasing brightness.

3.10. **Brunet–Derrida particle systems.** Berestycki and Zhuo Zhao have studied systems of N particles in two or more dimensions defined as follows. Let $s : \mathbb{R}^d \to \mathbb{R}$. Particles perform branching Brownian motion. At each reproduction time, the particle with the smallest s-score is removed from the system. The main result is that the particles form a cloud which travels at positive deterministic speed but in a possibly random direction. The dimension of the cloud scales like $\log N$ in the direction parallel to motion but at least $(\log N)^{3/2}$ in the orthogonal direction (and the conjecture is that this is sharp). This validates in part an argument often used by biologists to explain the pervasiveness of diploid populations.

3.11. **Random walks and random surfaces.** A natural family of random surfaces is obtained by allowing a random line to evolve in time according to a random walk. An instance of this construction, in a discrete setting, was analysed by Norris (with Boissard, Cohen, and Espinasse). Here, the random surface is specified by the constraint that the heights above neighbouring points of the planar lattice differ always by 1 and by the rule that all possible steps of the random walk are equally likely. Techniques from homogenization and analysis on graphs allow a quantitative identification of the diffusion limit behaviour of the random walk in terms of the number of steps in the random line.

3.12. **Sub-Riemannian diffusions.** The heat flow and diffusion process associated with a sub-Riemannian manifold show forms of behaviour which differ from the better understood Riemannian case. Norris (with Bailleul and Mesnager) is working to understand the Gaussian fluctuations for such diffusions in small time, when conditioned to bridge between two different points. The law concentrates near the path of minimal energy and we are able to characterise geometrically the deviation of the diffusion from this path.

3.13. **Martingale estimates for infinite-dimensional evolutions.** We are developing techniques to extract deterministic scaling limits from infinite-dimensional Markovian systems, using a combination of martingale estimates and finite-dimensional approximation. A recent application by Norris), not directly related to RaG, is a new and quantitative proof of the convergence of Kac's mean-field model for hard-sphere gas dynamics to the spatially homogeneous Boltzmann equation. In current work, Norris and Sola are investigating the application of these ideas to the Hasting–Levitov cluster evolutions discussed in Paragraph 3.1.

3.14. **Geometry of supercritical percolation.** It is a famous problem to show there is no infinite cluster in *critical* percolation, for general dimension. A related question is to decide whether the complement of the supercritical infinite cluster contains an infinite component. This is trivial for site percolation with $d \geq 3$ but highly non-trivial for bond percolation. Grimmett, Holroyd, and Kozma have answered the question affirmatively for large d, but the case $3 \leq d \leq 6$ remains open.

4. Activities

4.1. **Output.** The following publications and preprints have been facilitated by funding through RaG. They are available via

http://www.statslab.cam.ac.uk/~grg/rag-pubs.html

Preprints from this report period

1. Condensation of a two-dimensional random walk and the Wulff crystal, N. Berestycki, A. Yadin
2. The shape of multidimensional Brunet–Derrida particle systems, N. Berestycki, Lee Zhuo Zhao
3. Counting self-avoiding walks, G. Grimmett, Z. Li
4. Percolation of finite clusters and infinite surfaces, G. Grimmett, A. Holroyd, G. Kozma
5. Diffusion in planar Liouville quantum gravity, N. Berestycki
6. Cyclicity in Dirichlet-type spaces and extremal polynomials, C. Bénéteau, A. Condori, C. Liaw, D. Seco, A. Sola, *Journal d'Analyse Mathématique*
7. Expected discrepancy for zeros of random polynomials, I. Pritsker, A. Sola, *Proceedings of the American Mathematical Society*
8. Elementary examples of Loewner chains generated by densities, A. Sola, *Annales Universitatis Mariae Curie-Sklodowska*
9. Strict inequalities for connective constants of transitive graphs, G. Grimmett, Z. Li
10. Diffusivity of a random walk on random walks, E. Boissard, S. Cohen, T. Espinasse, J. Norris
11. Uniqueness of infinite homogeneous clusters in 1–2 model, Z. Li
12. Bounds on connective constants of regular graphs, G. Grimmett, Z. Li
13. Self-avoiding walks and the Fisher transformation, G. Grimmett, Z. Li
14. Influences in product spaces: BKKKL re-revisited, G. Grimmett, S. Janson, J. Norris, *arXiv*:1207.1780
15. Critical branching Brownian motion with absorption: particle configurations, J. Berestycki, N. Berestycki, J. Schweinsberg
16. Critical branching Brownian motion with absorption: survival probability, J. Berestycki, N. Berestycki, J. Schweinsberg

Publications and preprints from previous report periods

1. Three theorems in discrete random geometry, G. Grimmett. *Probability Surveys* 8 (2011) 403–441
2. A small-time coupling between Lambda-coalescents and branching processes, J. Berestycki, N. Berestycki, V. Limic, *Annals of Applied Probability*
3. The genealogy of branching Brownian motion with absorption, J. Berestycki, N. Berestycki, J. Schweinsberg, *Annals of Probability* 41 (2013) 527–618

4. Percolation since Saint-Flour, G. Grimmett, H. Kesten, in *Percolation Theory at Saint-Flour*, Springer, 2012, pages ix–xxvii
5. Cycle structure of the interchange process and representation theory, N. Berestycki, G. Kozma
6. Galton–Watson trees with vanishing martingale limit, N. Berestycki, N. Gantert, P. Moerters, N. Sidorova
7. Critical temperature of periodic Ising models, Z. Li
8. Spectral curve of periodic Fisher graphs, Z. Li
9. Bond percolation on isoradial graphs, G. Grimmett, I. Manolescu, *Probability Theory and Related Fields*
10. Asymptotic sampling formulae for Lambda-coalescents, J. Berestycki, N. Berestycki, V. Limic, *Ann. Inst. H. Poincaré B*
11. 1–2 model, dimers, and clusters, Z. Li
12. Large scale behaviour of the spatial Lambda–Fleming–Viot process, N. Berestycki, A. M. Etheridge, A. Veber, *Ann. Inst. H. Poincaré B* 49 (2013) 374–401
13. Hastings-Levitov aggregation in the small-particle limit, J. Norris, A. Turner, *Commun. Math. Phys.* (2012) 316, 809–841
14. Weak convergence of the localized disturbance flow to the coalescing Brownian flow, J. Norris, A. Turner, *Annals of Probability*
15. Universality for bond percolation in two dimensions, G. Grimmett, I. Manolescu, *Annals of Probability*
16. Inhomogeneous bond percolation on square, triangular, and hexagonal lattices, G. Grimmett, I. Manolescu, *Annals of Probability*
17. Cluster detection in networks using percolation, G. Grimmett, E. Arias-Castro, *Bernoulli* 19 (2013) 676–719

4.2. **Cambridge Conference.** A workshop on *Geometry and Analysis of Random Processes* took place under the auspices of RaG during the week 8–12 April 2013. It was co-funded by the European Science Foundation and Cambridge University. The programme, which can be found at

http://www.statslab.cam.ac.uk/~grg/EasterMeeting.html,

was based around mini-courses by O. Angel, O. Zeitouni, M. Hairer, and Grégory Miermont, complemented by a series of research talks by leading researchers, PhD students, and postdocs. Participation was markedly international, with a substantial UK participation of students, postdocs, and tenured staff in probability.

4.3. **Seminars.** The weekly probability seminar has been lively as always. Details of events may be found at

http://talks.cam.ac.uk/show/archive/9938.

4.4. **Visitors.** Cambridge Probability has received a number of visitors in 2012–13, for short and longer periods. The following individuals are connected directly to RaG.

- Jason Miller, April 2013
- Ander Holroyd, January 2013
- Omer Angel, April 2013
- Wendelin Werner, January–March 2013

The following have visited with non-RaG support.

- Jason Miller, November 2012
- Amanda Turner, December 2012
- Raphael Cerf, May–July 2013
- Svante Janson, June 2013

4.5. **Visits by members of RaG.** Members of RaG have made numerous visits to other institutions, and have participated in numerous conferences and workshops. Listed here are visits made by research fellows.

- Aug 2012: *Complex Analysis and Integrable Systems.* Institut Mittag-Leffler, Stockholm [Sola]
- Sep 2012: University of Sevilla, Sevilla. Research visit [Sola]
- Jan 2013: *Third Conference of Tsinghua Sanya International Mathematics Forum*, Sanya, China [Li]
- Jan 2013: Université Pierre et Marie Curie, Paris, France, Research visit [Li]
- Feb 2013: *Random Tilings*, Simons Center for Geometry and Physics, New York, USA [Li]
- Apr 2013: Columbia University, New York. Research visit [Sola]
- Apr 2013: University of South Florida, Tampa. Research visit [Sola]
- Apr 2013: *University of South Florida Math Colloquium*: Scaling limits in conformal aggregation models [Sola]
- Apr 2013: *Oklahoma State University Analysis Seminar*: Cyclicity in Dirichlet-type spaces: a concrete approach [Sola]

4.6. **Industrial outreach.** A discussion meeting took place on 19 July 2012 with scientists from the British Antarctic Survey.

5. Future activities

Amongst our immediate targets are the following.

- Our proposal for a 6 month programme at the Isaac Newton Institute, with N. Berestycki as principal organizer, has been accepted by the Institute, and planning is underway. It will take place in the first half of 2015.
- The search for and appointment of postdoctoral fellows within RaG.
- Planning for the next day of industrial outreach.

Statistical Laboratory, Centre for Mathematical Science, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB