# 3

# Critical Probabilities

## 3.1 Equalities and Inequalities

Let $G$ be a graph, and let $p_c(G)$ denote the critical probability of bond percolation on $G$, as in Section 1.6. It is tempting to seek an exact calculation of $p_c(G)$ for given $G$, but there seems no reason to expect a closed form for $p_c(G)$ unless $G$ has special structure. Indeed, except for certain famous two-dimensional lattices, the value of $p_c(G)$ may have no other special features. The exceptional cases include:

| | |
|---|---|
| square lattice | $p_c = \frac{1}{2}$ |
| triangular lattice | $p_c = 2\sin(\pi/18)$ |
| hexagonal lattice | $p_c = 1 - 2\sin(\pi/18)$ |
| bow-tie lattice | $p_c = p_c(\text{bow-tie})$ |

where $p_c(\text{bow-tie})$ is the unique root in $(0, 1)$ of the equation

$$1 - p - 6p^2 + 6p^3 - p^5 = 0.$$

See Figure 3.1 for drawings of these lattices.

It is the operation of 'duality' which is of primary value in establishing these exact values (the definition of planar dual is given beneath (1.16), see also Section 11.2). Given a planar lattice $\mathcal{L}$ (defined in an appropriate way not explored here) and its dual lattice $\mathcal{L}_d$, one may show that

$$p_c(\mathcal{L}) + p_c(\mathcal{L}_d) = 1 \tag{3.1}$$

subject to certain conditions of symmetry on $\mathcal{L}$. We do not present a proof of such a relation, since this would use techniques to be explored only later in this book. Equation (3.1) is equivalent to the following statement:

$$p > p_c(\mathcal{L}) \quad \text{if and only if} \quad 1 - p < p_c(\mathcal{L}_d),$$

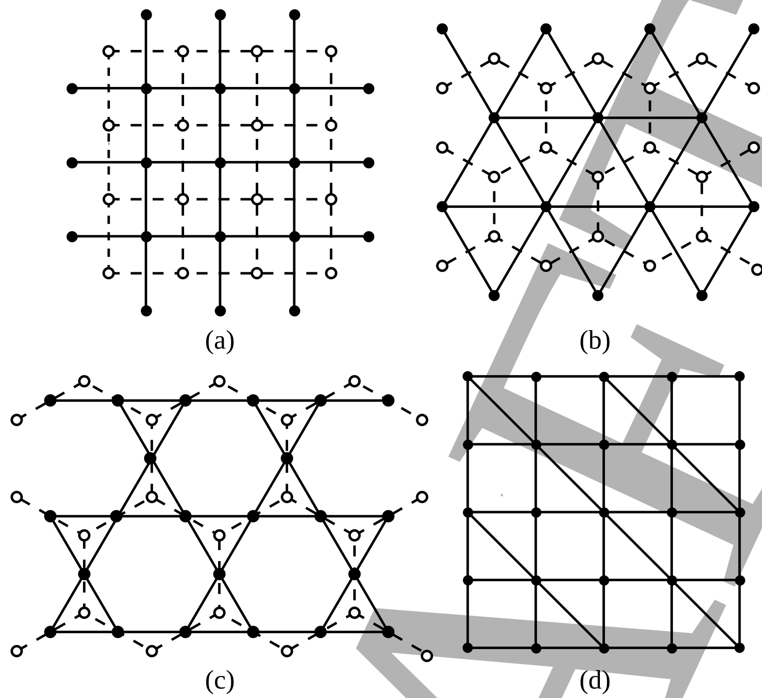


Figure 3.1. Five lattices in two dimensions. (a) The square lattice is self-dual. (b) The broken edges constitute the hexagonal (or 'honeycomb') lattice, and the solid edges constitute the triangular lattice, being the dual of the hexagonal lattice. (c) The broken edges constitute the hexagonal lattice, and the solid edges constitute the 'kagomé' lattice, being the covering lattice of the hexagonal lattice. (d) The so called bow-tie lattice.

for which an intuitive explanation is as follows. If $p > p_c(\mathcal{L})$, there exists (almost surely) an infinite open cluster of $\mathcal{L}$, and infinite clusters occupy a strictly positive density of space. If there is a unique such infinite cluster (which fact we shall prove in Chapter 8), then this cluster extends throughout space, and precludes the existence of an infinite closed cluster of $\mathcal{L}_d$; therefore $1 - p < p_c(\mathcal{L}_d)$. Conversely, if $p < p_c(\mathcal{L})$, all open clusters of $\mathcal{L}$ are (almost surely) finite, and the intervening space should contain an infinite closed cluster of $\mathcal{L}_d$; therefore, $1-p > p_c(\mathcal{L}_d)$. However appealing these crude arguments may be, their rigorous justification is highly non-trivial.

Once (3.1) is accepted, the exact value $p_c = \frac{1}{2}$ for the square lattice follows immediately, since this lattice is self-dual. A rigorous proof of this calculation appears in Section 11.3. When $\mathcal{L}$ is the triangular lattice, then $\mathcal{L}_d$ is the hexagonal lattice, and in this case we need another link between the two critical probabilities in order to compute them exactly. The so called 'star–triangle' relation provides such a link, and the exact values follow. See Section 11.9 for a complete derivation.

A similar argument is valid for the bow-tie lattice $\mathcal{L}$, namely that the dual of

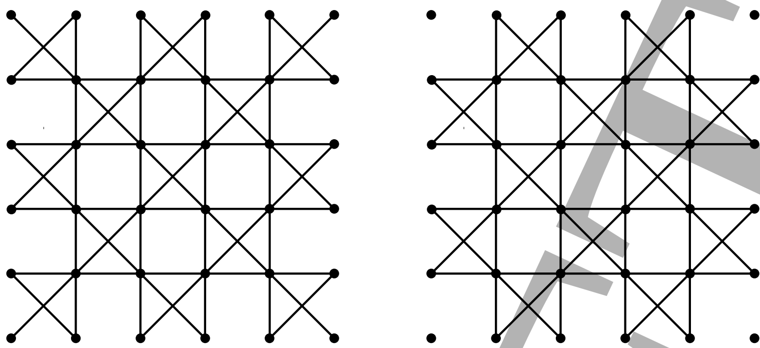

Figure 3.2. A graph $G$ may be used to generate a matching pair $\mathcal{G}_1$, $\mathcal{G}_2$. Any finite cluster of $\mathcal{G}_1$ is surrounded by a circuit of $\mathcal{G}_2$. In this picture, $G$ is the square lattice $\mathbb{L}^2$, and $\mathcal{G}_1$ and $\mathcal{G}_2$ are obtained by adding the diagonals to alternate faces of $G$. In this special case, both $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic to the covering lattice of $\mathbb{L}^2$.

$\mathcal{L}$ may be transformed into a copy of $\mathcal{L}$, by judicious use of the star–triangle transformation. This enables a computation of its critical value. There may exist other two-dimensional lattices to which similar arguments may be applied.

We turn now to site percolation. As observed in Section 1.6, the bond model on a graph $G$ is equivalent to the site model on the covering graph $G_c$. It follows in particular that the kagomé lattice, being the covering lattice of the hexagonal lattice, satisfies $p_c^{\text{site}}(\text{kagomé}) = 1 - 2\sin(\pi/18)$.

Whereas *duality* was a key to *bond* percolation in two dimensions, the corresponding property for *site* percolation is that of *matching*. A matching pair $\mathcal{G}_1$, $\mathcal{G}_2$ of graphs in two dimensions is constructed as follows. We begin with an infinite planar graph $G$ with 'origin' 0, and we select some arbitrary family $\mathcal{F}$ of faces of $G$. We obtain $\mathcal{G}_1$ (respectively $\mathcal{G}_2$) from $G$ by adding all diagonals to all faces in $\mathcal{F}$ (respectively all faces not in $\mathcal{F}$). The graphs $G$, $\mathcal{G}_1$, $\mathcal{G}_2$ have the same vertex sets, and therefore a site percolation process on $G$ induces site percolation processes on $\mathcal{G}_1$ and $\mathcal{G}_2$. If the origin 0 belongs to a finite open cluster of $\mathcal{G}_1$, then the external (vertex) boundary of this cluster forms a closed circuit of $\mathcal{G}_2$ (see the example in Figure 3.2). This turns out to be a very useful property. We say that $\mathcal{G}_1$ is *self-matching* if $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs. Note that, if $G$ is a triangulation (that is, if every face of $G$ is a triangle), then $G = \mathcal{G}_1 = \mathcal{G}_2$, and in this case $G$ is self-matching. The triangular lattice $\mathbb{T}$ is an example of a self-matching lattice. Further details and references concerning two-dimensional matching pairs may be found in Kesten (1982).

Let $\mathcal{G}_1$, $\mathcal{G}_2$ be a matching pair of lattices in two dimensions. Subject to assumptions on the pair $\mathcal{G}_1$, $\mathcal{G}_2$, one may on occasion be able to justify the relation

$$p_c^{\text{site}}(\mathcal{G}_1) + p_c^{\text{site}}(\mathcal{G}_2) = 1;$$

cf. (3.1). One may deduce that the triangular lattice $\mathbb{T}$, being self-matching, has

| | bond | site |
|---|---|---|
| hexagonal | | $\simeq 0.70$ |
| square $\mathbb{L}^2$ | | $\simeq 0.59$ |
| kagomé | $\simeq 0.52$ | |
| cubic $\mathbb{L}^3$ | $\simeq 0.25$ | $\simeq 0.31$ |

Table 3.1. Numerical estimates of critical probabilities. See Hughes (1996) for origins and explanations.

site critical probability $p_c^{\text{site}}(\mathbb{T}) = \frac{1}{2}$. Indeed, it is believed that $p_c^{\text{site}} = \frac{1}{2}$ for a broad family of 'reasonable' triangulations of the plane.

In the absence of a general method for computing critical percolation probabilities, we may have cause to seek inequalities. These come in two forms, rigorous and non-rigorous. A great deal of estimation of critical probabilities has been carried out, using a mixture of numerical, rigorous, and non-rigorous arguments. We do not survey such results here, but refer the reader to pages 182–183 of Hughes (1996). As an example of an inequality which is both rigorous and rather tight, Wierman (1990) has proved that

$$0.5182 \le p_c^{\text{bond}}(\text{kagomé}) \le 0.5335,$$

but other results of this type are generally rather weak.

Another line of enquiry has been to understand the behaviour of critical probabilities in the limit as the number $d$ of dimensions is allowed to pass to infinity. We shall encounter in Section 10.3 the technology known as the 'lace expansion', which has been developed by Hara and Slade (1990, 1994, 1995) in order to understand percolation when $d$ is large and finite. When applied to bond percolation on $\mathbb{L}^d$, these arguments imply an expansion of which the first terms follow:

$$(3.2) \qquad p_c^{\text{bond}}(\mathbb{L}^d) = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{7/2}{(2d)^3} + \mathrm{O}\left(\frac{1}{(2d)^4}\right) \quad \text{as } d \to \infty.$$

The remainder of this chapter is devoted to a method for proving strict inequalities *between* critical probabilities. This method appears to have fundamental merit in situations where one needs to understand whether a systematic addition of edges to a process causes a *strict* change in its critical value. In Section 3.2 is presented an example of this argument at work; see Theorem (3.7). Section 3.3 contains a general formulation of enhancements for percolation models. Such methods are adapted in Section 3.4 to obtain strict inequalities between site and bond critical probabilities of cubic lattices.

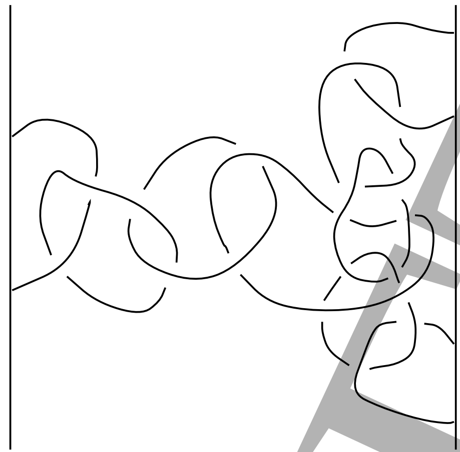

Figure 3.3. An entanglement between opposite sides of a cube in three dimensions. Note the chain of necklaces on the right.

## 3.2 Strict Inequalities

If $\mathcal{L}$ is a sublattice of the lattice $\mathcal{L}'$ (written $\mathcal{L} \subseteq \mathcal{L}'$) then clearly their critical probabilities satisfy $p_c(\mathcal{L}) \geq p_c(\mathcal{L}')$, since any infinite open cluster of $\mathcal{L}$ is contained in some infinite open cluster of $\mathcal{L}'$. When does the *strict* inequality $p_c(\mathcal{L}) > p_c(\mathcal{L}')$ hold? The question may be quantified by asking for non-trivial lower bounds for $p_c(\mathcal{L}) - p_c(\mathcal{L}')$.

Similar questions arise in many ways, not simply within percolation theory. More generally, consider any process indexed by a continuously varying parameter $T$ and enjoying a phase transition at some critical point $T = T_c$. In many cases of interest, sufficient structure is available to enable the conclusion that certain systematic changes to the process can only change $T_c$ in one particular direction. For example, one may be able to conclude that the critical value of the altered process is no greater than $T_c$. The question then is to understand which systematic changes decrease $T_c$ *strictly*. In the context of the previous paragraph, the systematic changes in question may involve the 'switching on' of edges lying in $\mathcal{L}'$ but not in $\mathcal{L}$.

A related percolation question is that of 'entanglements'. Consider bond percolation on $\mathbb{L}^3$, and examine the box $B(n)$. We think of the open edges as being solid connections made of elastic, say, and we may try to 'pull apart' a pair of opposite faces of $B(n)$. If $p > p_c$, we will generally fail because, with large probability (tending to 1 as $n \to \infty$), there exists an open path joining one face to the opposite face. We may fail even if $p < p_c$, owing to an 'entanglement' of open paths (a chain of necklaces, perhaps, see Figure 3.3). It may be seen that there exists an 'entanglement transition' at some critical point $p_c^{\text{ent}}$ satisfying $p_c^{\text{ent}} \leq p_c$. Is it the case that strict inequality holds, that is, that $p_c^{\text{ent}} < p_c$?

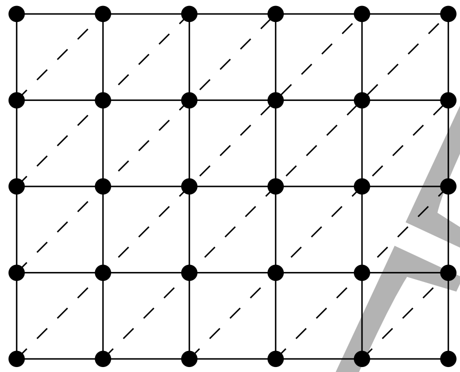

Figure 3.4. The triangular lattice may be obtained from the square lattice by the addition of certain diagonals.

A technology has been developed for approaching such questions of strict inequality. Although, in particular cases, ad hoc arguments can be successful, there appears to be only one general approach known currently. We illustrate this approach next, by sketching the details in a particular case. A more general argument will be presented in Section 3.3, and this will allow an answer to the entanglement question above.

The triangular lattice $\mathbb{T}$ may be obtained by adding diagonals across the squares of the square lattice $\mathbb{L}^2$, in the manner of Figure 3.4. Since any infinite open cluster of $\mathbb{L}^2$ is contained in an infinite open cluster of $\mathbb{T}$, it follows that $p_c(\mathbb{T}) \le p_c(\mathbb{L}^2)$, but does strict inequality hold? There are various ways of showing that the answer is affirmative. Here we adopt the canonical argument of Aizenman and Grimmett (1991), as an illustration of a general technique. The reason for including this special case in advance of the more general formulation of Theorem (3.16) is that it illustrates clearly the structure of the method with a minimum of complications.

We point out that, *for this particular case,* there is a variety of ways of obtaining the result, by using special properties of the square and triangular lattices. The attraction of the method described here is its generality, relying as it does on essentially no assumptions about lattice structure or number of dimensions.

First we embed the problem in a two-parameter system. Let $p, s \in [0,1]^2$. We declare each edge of $\mathbb{L}^2$ to be open with probability $p$, and each *further edge* of $\mathbb{T}$ (that is, the dashed edges in Figure 3.4) to be open with probability $s$. Writing $P_{p,s}$ for the associated measure, define

$$\theta(p, s) = P_{p,s}(0 \leftrightarrow \infty). \tag{3.3}$$

We propose to establish certain differential inequalities which will imply that $\partial\theta/\partial p$ and $\partial\theta/\partial s$ are comparable, uniformly on any closed subset of the interior $(0,1)^2$ of the parameter space. This cannot itself be literally achieved, since we have insufficient information about the differentiability of $\theta$. Therefore we shall

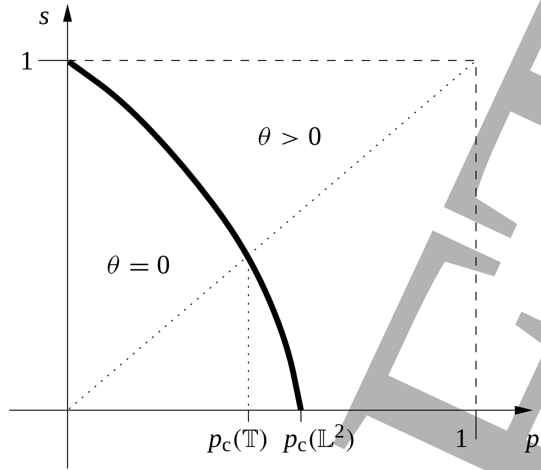


Figure 3.5. The 'critical curve'. The area beneath the curve is the set of $(p, s)$ for which $\theta(p, s) = 0$.

approximate $\theta$ by a finite-volume quantity $\theta_n$, and shall work with the partial derivatives of $\theta_n$.

Let $B(n) = [-n, n]^d$, and define

$$\theta_n(p, s) = P_{p,s}\big(0 \leftrightarrow \partial B(n)\big). \tag{3.4}$$

Note that $\theta_n$ is a polynomial in $p$ and $s$, and that

$$\theta_n(p, s) \downarrow \theta(p, s) \qquad \text{as } n \to \infty.$$

**(3.5) Lemma.** *There exist a positive integer L and a continuous function $\alpha$ mapping $(0, 1)^2$ to $(0, \infty)$ such that*

$$\alpha(p, s)^{-1} \frac{\partial}{\partial p}\theta_n(p, s) \geq \frac{\partial}{\partial s}\theta_n(p, s) \geq \alpha(p, s)\frac{\partial}{\partial p}\theta_n(p, s) \tag{3.6}$$

*for $0 < p, s < 1$ and $n \geq L$.*

Once this is proved, the main result follows immediately, namely the following.

**(3.7) Theorem.** *It is the case that $p_c(\mathbb{T}) < p_c(\mathbb{L}^2)$.*

**Proof of Theorem (3.7).** Here is a rough argument, the rigour comes later. It may be shown that there exists a 'critical curve' in $(p, s)$-space, separating the regime where $\theta(p, s) = 0$ from that when $\theta(p, s) > 0$ (see Figure 3.5). Suppose that this critical curve may be written in the form $h(p, s) = 0$ for some increasing and continuously differentiable function $h$ satisfying $h(p,s) = \theta(p,s)$ whenever $\theta(p,s) > 0$. It is enough to prove that the critical curve contains no vertical segment, and we shall prove this by working with the gradient vector

$$\nabla h = \left( \frac{\partial h}{\partial p}, \frac{\partial h}{\partial s} \right).$$

We take some liberties with (3.6) in the limit as $n \to \infty$, and deduce that

$$\nabla h \cdot (0,1) = \frac{\partial h}{\partial s} \geq \alpha(p,s) \frac{\partial h}{\partial p},$$

whence

$$\frac{1}{|\nabla h|} \frac{\partial h}{\partial s} = \left\{ \left( \frac{\partial h}{\partial p} \Big/ \frac{\partial h}{\partial s} \right)^2 + 1 \right\}^{-\frac{1}{2}} \geq \frac{\alpha}{\sqrt{\alpha^2 + 1}},$$

which is bounded away from 0 on any closed subset of $(0,1)^2$. This indicates as required that the critical curve has no vertical segment.

Here is the proper argument. Let $\eta$ be positive and small, and find $\gamma$ $(> 0)$ such that $\alpha(p,s) \geq \gamma$ on $[\eta, 1-\eta]^2$. Let $\psi \in [0, \pi/2)$ satisfy $\tan\psi = \gamma^{-1}$.

At the point $(p,s) \in [\eta, 1-\eta]^2$, the rate of change of $\theta_n(p,s)$ in the direction $(\cos\psi, -\sin\psi)$ satisfies

$$\text{(3.8)} \qquad \begin{aligned} \nabla\theta_n \cdot (\cos\psi, -\sin\psi) &= \frac{\partial\theta_n}{\partial p} \cos\psi - \frac{\partial\theta_n}{\partial s} \sin\psi \\ &\leq \frac{\partial\theta_n}{\partial p} (\cos\psi - \gamma \sin\psi) = 0 \end{aligned}$$

by (3.6), since $\tan\psi = \gamma^{-1}$.

Suppose now that $(a,b) \in [2\eta, 1-2\eta]^2$. Let

$$(a',b') = (a,b) + \eta(\cos\psi, -\sin\psi),$$

noting that $(a',b') \in [\eta, 1-\eta]^2$. By integrating (3.8) along the line segment joining $(a,b)$ to $(a',b')$, we obtain that

$$\text{(3.9)} \qquad \theta(a',b') = \lim_{n\to\infty} \theta_n(a',b') \leq \lim_{n\to\infty} \theta_n(a,b) = \theta(a,b).$$

There is quite a lot of information in such a calculation, but we abstract a small amount only. Let $\eta$ be small and positive. Take $(a,b) = (p_c(\mathbb{T}) - \zeta, p_c(\mathbb{T}) - \zeta)$ and define $(a',b')$ as above. We choose $\zeta$ sufficiently small that $(a,b), (a',b') \in [2\eta, 1-2\eta]^2$, and that $a' > p_c(\mathbb{T})$. The above calculation implies that

$$\text{(3.10)} \qquad \theta(a',0) \leq \theta(a',b') \leq \theta(a,b) = 0,$$

whence $p_c(\mathbb{L}^2) \geq a' > p_c(\mathbb{T})$. □

**Proof of Lemma (3.5).** With $\mathbb{E}^2$ the edge set of $\mathbb{L}^2$, and $\mathbb{F}$ the additional edges in the triangular lattice $\mathbb{T}$ (that is, the diagonals in Figure 3.4), we have by Russo's formula (in a slightly more general version than Theorem 2.25) that

$$\begin{aligned} \frac{\partial}{\partial p}\theta_n(p,s) &= \sum_{e\in\mathbb{E}^2} P_{p,s}(e \text{ is pivotal for } A_n), \\ \frac{\partial}{\partial s}\theta_n(p,s) &= \sum_{f\in\mathbb{F}} P_{p,s}(f \text{ is pivotal for } A_n), \end{aligned} \tag{3.11}$$

where $A_n = \{0 \leftrightarrow \partial B(n)\}$. The idea now is to show that each summand in the first summation is comparable with some given summand in the second. Actually we shall only prove the second inequality in (3.6), since this is the only one used in proving the above theorem, and in addition the proof of the other part is similar.

With each edge $e$ of $\mathbb{E}^2$ we associate a unique edge $f = f(e)$ of $\mathbb{F}$ such that $f$ lies near to $e$. This may be done in a variety of ways, but in order to be concrete we specify that if $e = \langle z, z+u_1\rangle$ or $e = \langle z, z+u_2\rangle$ then $f = \langle z, z+u_1+u_2\rangle$, where $u_1$ and $u_2$ are unit vectors in the directions of the (increasing) $x$ and $y$ axes respectively.

We claim that there exists a function $\beta(p,s)$, continuous on $(0,1)^2$, such that, for all sufficiently large $n$:

$$P_{p,s}\big(e \text{ is pivotal for } A_n\big) \leq \beta(p,s) P_{p,s}\big(f(e) \text{ is pivotal for } A_n\big) \tag{3.12}$$

for all $e$ lying in $B(n)$. Once this is shown, we sum over $e$ to obtain by (3.11) that

$$\begin{aligned} \frac{\partial}{\partial p}\theta_n(p,s) &\leq \beta(p,s) \sum_{e\in\mathbb{E}^2} P_{p,s}\big(f(e) \text{ is pivotal for } A_n\big) \\ &\leq 2\beta(p,s) \sum_{f\in\mathbb{F}} P_{p,s}(f \text{ is pivotal for } A_n) \\ &= 2\beta(p,s)\frac{\partial}{\partial s}\theta_n(p,s) \end{aligned}$$

as required. The factor 2 arises because, for each $f$ ($\in \mathbb{F}$), there are exactly two edges $e$ with $f(e) = f$.

The idea of the proof of (3.12) is that, if $e$ is pivotal for $A_n$ in the configuration $\omega$, then, by making 'local changes' to $\omega$, we may create a configuration in which $f(e)$ is pivotal for $A_n$. The factor $\beta$ in (3.12) reflects the cost of making such a local change.

Here is a fairly formal proof of (3.12). Suppose that $e = \langle z, z+u_1\rangle$ where $u_1 = (1,0)$; a similar argument will be valid with $u_1$ replaced by $u_2 = (0,1)$. Let $B_e = z + B(2)$, a box centred at $z$, and let $\mathbb{E}_e$ be the set of edges of $\mathbb{T}$ having at

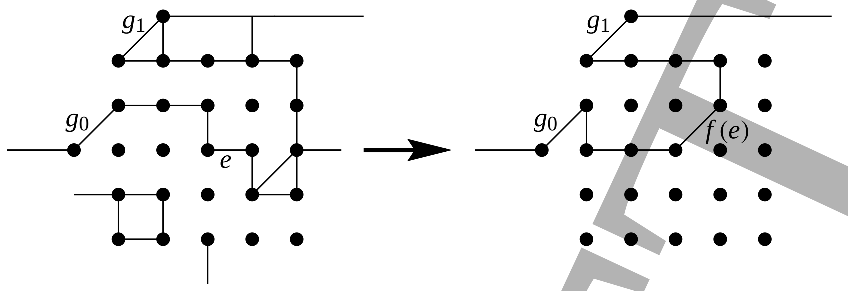


Figure 3.6. An example of a configuration $\omega$ on $\mathbb{E}_e$ which gives rise to a configuration $\omega' = \omega'(e, \omega)$.

least one vertex in $B_e$. Suppose for the moment that $0 \notin B_e$ and $B_e \cap \partial B(n) = \varnothing$. Let $\omega$ be a configuration in which $e$ is pivotal for $A_n$. If $e$ were open, then all paths from 0 to $\partial B(n)$ would by necessity pass along $e$. Therefore, there exist two edges $g_i = \langle a_i, b_i \rangle$ of $\mathbb{T}$ (for $i = 1, 2$) such that:

(i) $a_i \in \partial B_e$, $b_i \notin B_e$, and the edge $\langle a_i, b_i \rangle$ is open, for $i = 1, 2$,

(ii) in the configuration obtained from $\omega$ by declaring all edges in $\mathbb{E}_e \setminus \{g_0, g_1\}$ to be closed, we have that $0 \leftrightarrow a_0$ and $\partial B(n) \leftrightarrow a_1$.

If there is a choice for the edges $g_i$ then we pick them according to some predetermined ordering of all edges. See Figure 3.6.

Having found the $g_i$, we may find a configuration $\omega'$ ($\in \Omega$) such that:

(iii) $\omega$ and $\omega'$ agree off $\mathbb{E}_e \setminus \{g_0, g_1\}$,

(iv) $\omega' \in \{f(e)$ is pivotal for $A_n\}$.

The idea in the construction of $\omega'$ is to find two vertex-disjoint paths $\pi_0$ and $\pi_1$ of $\mathbb{T}$ having vertices in $B_e$, and such that $\pi_0$ joins $a_0$ to $z$, and $\pi_1$ joins $a_1$ to $z+u_1+u_2$; then we define $\omega'$ by

$$\omega'(h) = \begin{cases} \omega(h) & \text{if } h \notin \mathbb{E}_e \setminus \{g_0, g_1\}, \\ 1 & \text{if } h \text{ lies in } \pi_0 \text{ or } \pi_1, \\ 1 & \text{if } h = f(e), \\ 0 & \text{otherwise.} \end{cases}$$

This construction is illustrated further in Figure 3.7. It may be seen from the figures that $\omega'$ satisfies (iv) above. We write $\omega' = \omega'(e, \omega)$ to emphasize the dependence of $\omega'$ on the choice of $e$ and $\omega$.

If $e$ is such that either $0 \in B_e$ or $B_e \cap \partial B(n) \neq \varnothing$, then one may find a configuration $\omega'$ satisfying (iii) and (iv), although a slightly different geometrical construction is needed for these special cases. We omit the details of this, noting only the conclusion that, for each $e$ and $\omega \in \{e$ is pivotal for $A_n\}$, there exists $\omega'$ satisfying (iii) and (iv) above. It follows from (iii) that

$$P_{p,s}(\omega) \le \frac{1}{\gamma^R} P_{p,s}(\omega')$$

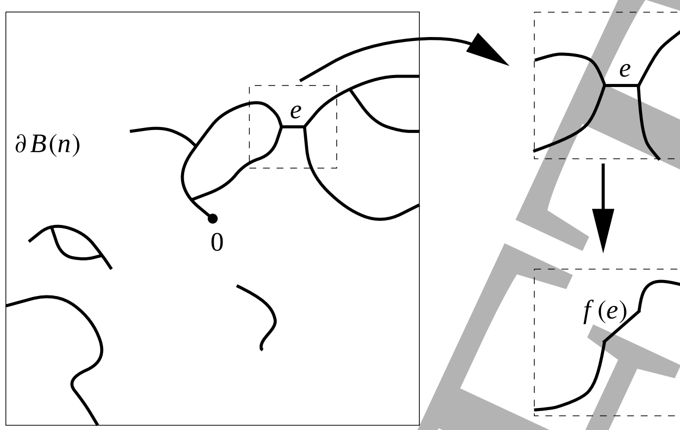


Figure 3.7. Inside the box $B(n)$, the edge $e$ is pivotal for the event $\{0 \leftrightarrow \partial B(n)\}$. By altering the configuration inside the smaller box, we may construct a configuration in which $f(e)$ is pivotal.

where $\gamma = \min\{p, s, 1-p, 1-s\}$ and $R = |\mathbb{E}_e|$.

Write $E_h$ for the event that an edge $h$ is pivotal for $A_n$. For $\omega \in E_e$, we have by (iv) that $\omega' \in E_{f(e)}$. Therefore,

$$P_{p,s}(E_e) = \sum_{\omega \in E_e} P_{p,s}(\omega) \le \sum_{\omega \in E_e} \frac{1}{\gamma^R} P_{p,s}(\omega') \le \left(\frac{2}{\gamma}\right)^R P_{p,s}(E_{f(e)}),$$

and (3.12) follows with $\beta(p,s) = (2/\gamma)^R$. □

## 3.3 Enhancements

An ‘enhancement’ is defined loosely to be a systematic addition of connections according to local rules. Enhancements may involve further coin flips. Can an enhancement create an infinite cluster when previously there was none?

Clearly the answer can be negative. For example, the enhancement may be of the type: join any two neighbours of $\mathbb{L}^d$ with probability $\frac{1}{2}p_c$ whenever they have no incident open edges. Such an enhancement creates extra connections but creates (almost surely) no extra infinite cluster.

Here is a proper definition of the concept of enhancement for bond percolation on $\mathbb{L}^d$ with parameter $p$. Let $R$ be a positive integer, and let $\mathcal{G}$ be the set of all simple graphs on the vertex set $B = B(R)$. Note that the set of open edges of any configuration $\omega$ $(\in \Omega)$ generates a member of $\mathcal{G}$, denoted $\omega_B$; $\mathcal{G}$ contains in addition many other graphs. Let $F$ be a function which associates with every $\omega_B$ a graph in $\mathcal{G}$. We call $R$ the 'enhancement range' and $F$ the 'enhancement function'. In the remainder of this chapter, we denote by $e + x$ the translate of an edge $e$ by the vector $x$; similarly, $G + x$ denotes the translate by $x$ of the graph $G$ on the vertex set $\mathbb{Z}^d$.

We shall consider making an enhancement at each vertex $x$ of $\mathbb{L}^d$, and we shall do this in a stochastic fashion. To this end, we provide ourselves with a vector $\eta = (\eta(x) : x \in \mathbb{Z}^d)$ lying in the space $\Xi = \{0,1\}^{\mathbb{Z}^d}$. We shall interpret the value $\eta(x) = 1$ as meaning that the enhancement at the vertex $x$ is 'activated'.

These ideas are applied in the following way. For each $x \in \mathbb{Z}^d$, we observe the configuration $\omega$ on the box $x + B$, and we write $F(x, \omega)$ for the associated evaluation of $F$. That is to say, we set $F(x, \omega) = F((\tau_x \omega)_B)$ where $\tau_x$ is the shift operator on $\Omega$ given by $\tau_x \omega(e) = \omega(e + x)$. The enhanced configuration is defined to be the graph

$$G^{\text{enh}}(\omega, \eta) = G(\omega) \cup \left\{ \bigcup_{x:\eta(x)=1} \{x + F(x, \omega)\} \right\} \tag{3.13}$$

where $G(\omega)$ is the graph of open edges under $\omega$. In writing the union of graphs, we mean the graph with vertex set $\mathbb{Z}^d$ having the union of the appropriate edge sets; wherever this union contains two or more edges between the same pair of vertices, these edges are allowed to coalesce into a single edge.

Thus we associate with each pair $(\omega, \eta) \in \Omega \times \Xi$ an enhanced graph $G^{\text{enh}}(\omega, \eta)$. We endow the sample space $\Omega \times \Xi$ with the product probability measure $P_{p,s}$, and we refer to the parameter $s$ as the *density* of the enhancement.

We call the enhancement function $F$ *essential* if there exists a configuration $\omega$ ($\in \Omega$) such that $G(\omega) \cup F(\omega)$ contains a doubly-infinite path but $G(\omega)$ contains no such path. Here are two examples of this definition.

(i) Suppose that $F$ has the effect of adding an edge joining the origin and any given unit vector whenever these two vertices are isolated in $G(\omega)$. Then $F$ is not essential.

(ii) If, on the other hand, $F$ adds such an edge whether or not the endvertices are isolated, then $F$ is indeed essential.

We call the enhancement function $F$ *monotonic* if, for all $\eta$ and all $\omega \le \omega'$, the graph $G^{\text{enh}}(\omega, \eta)$ is a subgraph of $G^{\text{enh}}(\omega', \eta)$. For $F$ to be monotonic it suffices that $\omega_B \cup F(\omega_B)$ be a subgraph of $\omega'_B \cup F(\omega'_B)$ whenever $\omega \le \omega'$.

The *enhanced percolation probability* is defined as

$$\theta^{\text{enh}}(p, s) = P_{p,s}\big(0 \text{ belongs to an infinite cluster of } G^{\text{enh}}\big). \tag{3.14}$$

A useful definition of the *enhancement critical point* is given by

$$p_c^{\text{enh}}(F, s) = \inf\{p : \theta^{\text{enh}}(p, s) > 0\}. \tag{3.15}$$

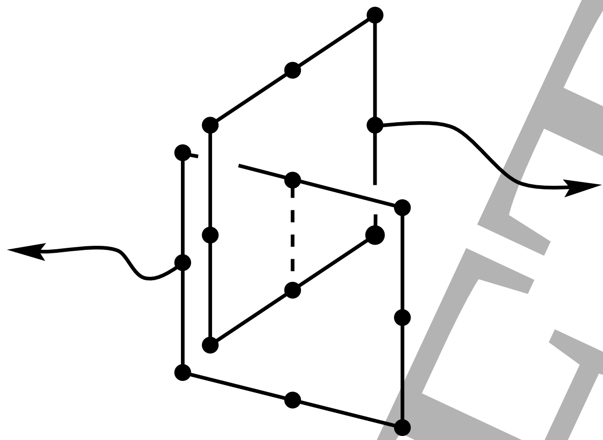

Figure 3.8. A sketch of the enhancement which adds an edge between any two interlocking $2 \times 2$ squares in $\mathbb{L}^3$. This picture contains a doubly-infinite path if and only if the enhancement is activated.

We note from (3.13) that $\theta^{\text{enh}}$ is non-decreasing in $s$. If $F$ is monotonic then, by Theorem (2.1), $\theta^{\text{enh}}$ is non-decreasing in $p$ also, whence

$$\theta^{\text{enh}}(p, s)\begin{cases} = 0 & \text{if } p < p_{\text{c}}^{\text{enh}}(F, s), \\ > 0 & \text{if } p > p_{\text{c}}^{\text{enh}}(F, s). \end{cases}$$

If $F$ is not monotonic, there will generally be ambiguity over the correct definition of the critical point. We will abide by (3.15) here.

**(3.16) Theorem.** *Let $s > 0$. If the enhancement function $F$ is essential, then $p_{\text{c}}^{\text{enh}}(F, s) < p_{\text{c}}$.*

The main point is that essential enhancements shift the critical point *strictly*. Instead of enhancements, one may study ‘diminishments’, which involve the systematic removal of open edges according to some local rule. A similar theorem may then be formulated, asserting that the critical point is strictly increased so long as the diminishment in question satisfies a condition parallel to that given above.

Here are some examples of Theorem (3.16) and related arguments.

A. *Entanglements.* Consider bond percolation on the three-dimensional cubic lattice $\mathbb{L}^3$. Whenever we see two interlinking $2 \times 2$ open squares, we join them by an edge (see Figure 3.8). It is easy to see that this enhancement is essential, and therefore it shifts the critical point downwards. Any reasonable definition of entanglement would require that two such interlocking squares be entangled, and it would follow that $p_{\text{c}}^{\text{ent}} < p_{\text{c}}$. We do not formulate precisely the notion of an entanglement since there are certain difficulties over this; see Section 12.5.

B. *Lattices and sublattices.* Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{L}'$. Assuming a reasonable definition of the term 'lattice', there will exist a periodic class $\mathcal{E}$ of edges of $\mathcal{L}'$ which do not lie in $\mathcal{L}$. Suppose it is the case that each $e \in \mathcal{E}$ is such that: there exists a bond configuration on $\mathcal{L}$ containing no doubly-infinite open path, but such a path exists if we add $e$ to the configuration. Although Theorem (3.16) cannot be applied directly in this situation, its proof may be adapted in a straightforward manner to deduce (rather as in Section 3.2) that $p_c(\mathcal{L}) > p_c(\mathcal{L}')$.

C. *Slabs* Let $d \geq 3$, and define the slab $S_k$ of thickness $k$ by $S_k = \mathbb{Z}^2 \times \{0, 1, 2, \dots, k\}^{d-2}$ where $k \geq 0$. Since $S_k \subseteq S_{k+1}$, we have that $p_c(S_k) \geq p_c(S_{k+1})$. The method of Theorem (3.16) may be used as follows to obtain the strict inequality $p_c(S_k) > p_c(S_{k+1})$. Let $e$ be the unit vector $(0, 0, \dots, 0, 0, 1)$. Take $\mathcal{L}'$ to be the graph derived from $\mathbb{L}^d$ by deleting all edges of the form $\langle x, x+e\rangle$ such that $|x_d + 1|$ is divisible by $k+2$. We construct the subgraph $\mathcal{L}$ of $\mathcal{L}'$ by deleting all edges $\langle x, x+e\rangle$ such that $|x_d + 1|$ is divisible by $k+1$. Then $\mathcal{L}'$ may be obtained by systematic enhancements of $\mathcal{L}$, and the claim may now be obtained in the usual way.

D. *Augmented percolation.* Here is a question which has arisen in so called 'invasion percolation'. Consider bond percolation on a lattice $\mathcal{L}$. Each edge is in exactly one of three categories: (i) open, (ii) closed and belonging to a finite closed cluster, (iii) closed and belonging to an infinite closed cluster. Consider the graph obtained from $\mathcal{L}$ by deleting all edges lying in category (iii) while retaining those in categories (i) and (ii). Does there exist an interval of values of $p$ ($< p_c$) for which this graph contains (almost surely) an infinite component? That this indeed holds for $\mathbb{L}^d$ with $d \geq 2$ follows by considering the enhancement in which an edge is added between the origin and a neighbour $x$ if and only if all other edges incident to 0 and $x$ are open.

E. *Site percolation.* The proof of Theorem (3.16) may be adapted to bond and site percolation on general lattices. The condition of 'essentialness' was formulated above for bond percolation, and is replaced as follows for site percolation. We say that the realization $\xi \in \{0,1\}^{\mathbb{Z}^d}$ of site percolation contains a doubly-infinite *self-repelling* path if there exists a doubly-infinite open path none of whose vertices is adjacent to any other vertex of the path except for its two neighbours in the path. An enhancement of site percolation is called *essential* if there exists a configuration $\xi$ containing no doubly-infinite self-repelling path, but such that the enhanced configuration obtained by activating the enhancement at the origin does indeed contain such a path.

**Proof of Theorem (3.16).** We follow Aizenman and Grimmett (1991). In this proof we shall construct various functions on $(0, 1)^2$, denoted as $\delta_i$ for $i \geq 1$. Such functions shall by convention be continuous and strictly positive on their domain $(0, 1)^2$.

The first step is to generalize equations (3.11). A pair $(\omega, \eta) \in \Omega \times \Xi$ gives rise to an enhanced graph $G^{\text{enh}}(\omega, \eta)$, and we call the edges of this graph *enhanced*. For $(\omega, \eta) \in \Omega \times \Xi$ and $e \in \mathbb{E}^d$, $x \in \mathbb{Z}^d$, we define configurations $\omega^e$, $\omega_e$, $\eta^x$, $\eta_x$ by

$$\omega^e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 1 & \text{if } f = e, \end{cases} \qquad \omega_e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 0 & \text{if } f = e, \end{cases}$$
$$\eta^x(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ 1 & \text{if } y = x, \end{cases} \qquad \eta_x(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ 0 & \text{if } y = x. \end{cases}$$

Let $n$ be a positive integer, and let $A = A_n$ be the event that there exists a path of enhanced edges joining the origin to some vertex of the set $\partial B(n)$. For $(\omega, \eta) \in \Omega \times \Xi$ and $e \in \mathbb{E}^d$, $x \in \mathbb{Z}^d$, we say that

$$\begin{aligned} &e \text{ is } (+)\text{pivotal for } A \text{ if } I_A(\omega^e, \eta) = 1 \text{ and } I_A(\omega_e, \eta) = 0, \\ &e \text{ is } (-)\text{pivotal for } A \text{ if } I_A(\omega^e, \eta) = 0 \text{ and } I_A(\omega_e, \eta) = 1, \\ &x \text{ is } (+)\text{pivotal for } A \text{ if } I_A(\omega, \eta^x) = 1 \text{ and } I_A(\omega, \eta_x) = 0, \end{aligned}$$

where
$$I_A(\omega, \eta) = \begin{cases} 1 & \text{if } A \text{ occurs}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that, if the enhancement is not monotonic, edges may generally be $(-)$pivotal for an increasing event $A$. Vertices, on the other hand, can only be $(+)$pivotal for an increasing event.

Since the occurrence of $A$ depends on only finitely many of the $\omega(e)$ and $\eta(x)$, we have by a minor extension of Theorem (2.32) that $\theta_n(p, s) = P_{p,s}(A)$ satisfies

$$\begin{aligned} \frac{\partial \theta_n}{\partial p} &= \sum_{e \in \mathbb{E}^d} \Big\{ P_{p,s}\big(e \text{ is } (+)\text{pivotal for } A\big) - P_{p,s}\big(e \text{ is } (-)\text{pivotal for } A\big) \Big\} \\ \frac{\partial \theta_n}{\partial s} &= \sum_{x \in \mathbb{Z}^d} P_{p,s}\big(x \text{ is } (+)\text{pivotal for } A\big). \end{aligned} \tag{3.17}$$

We continue with a geometrical observation. Recall that $R$ is the range of the enhancement. Let $m$ be a positive integer satisfying $m > R + 2$, and let $v$, $w$ be distinct vertices in $\partial B(m)$. The enhancement has been assumed essential, which is to say that there exists a bond configuration $\omega$ having the following property: $\omega$ contains no doubly-infinite open path, but such a path $\pi = \pi(\omega)$ is created when the enhancement at the origin is activated. Such a path $\pi$ must contain two disjoint singly-infinite open paths of $\omega$, denoted $\pi_1 = x_0, f_0, x_1, f_1, \dots$ and $\pi_2 = y_0, g_0, y_1, g_1, \dots$, such that $x_0, y_0 \in B(R)$. Let

$$r = \min\big\{i : x_i \in \partial B(m-1)\big\}, \quad s = \min\big\{i : y_i \in \partial B(m-1)\big\},$$

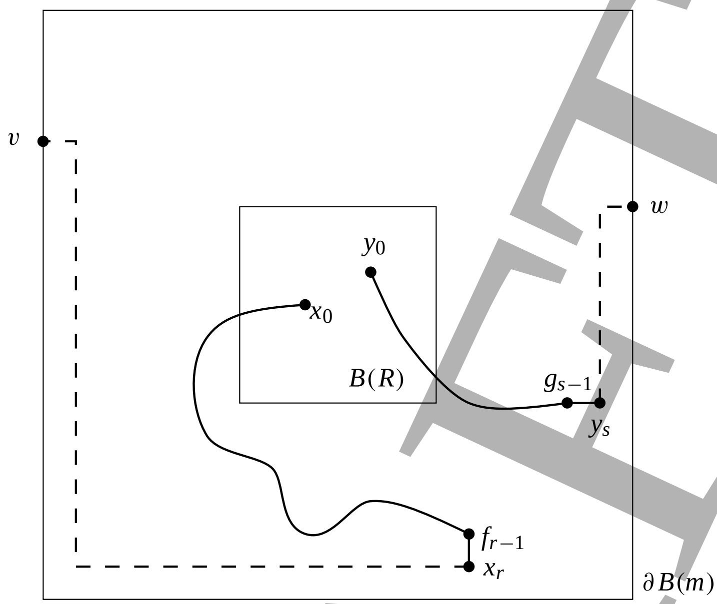


Figure 3.9. An illustration of the way in which $\widehat{\omega}$ is constructed when the enhancement is essential.

noting that $r, s \geq 2$. We may find vertex-disjoint paths $\nu_1$, $\nu_2$ of $\mathbb{L}^d$, using vertices in $B(m) \setminus B(m-2)$ only, such that $\nu_1$ joins $x_r$ to $v$ and $\nu_2$ joins $y_s$ to $w$. We now define the configuration $\widehat{\omega}$ $(\in \Omega)$ by:

$$\widehat{\omega}(e) = \begin{cases} \omega(e) & \text{if } e \text{ has both its endvertices in } B(m-2), \\ 1 & \text{if } e \text{ lies in either } \nu_1 \text{ or } \nu_2, \\ 1 & \text{if } e = f_{r-1}, g_{s-1}, \\ 0 & \text{otherwise.} \end{cases}$$

About $\widehat{\omega}$ we note the following:

(i) all open edges of $\widehat{\omega}$ have both endvertices in $B(m)$,

(ii) $\widehat{\omega}$ has no open path joining $v$ and $w$,

(iii) if the enhancement at 0 is activated, then an enhanced path is created joining $v$ and $w$ and using vertices of $B(m)$ only.

We write $\widehat{\omega} = \widehat{\omega}_m(v, w)$ to emphasize the dependence of $\widehat{\omega}$ on $n$, $v$, $w$. The construction of $\widehat{\omega}$ is illustrated in Figure 3.9.

Suppose that $e = \langle z, z+u \rangle$ where $u$ is a (positive) unit vector of $\mathbb{L}^d$. Let $B_e = z + B(m+R)$, a box centred at $z$, and let $v_1, v_2, \dots$ be a fixed ordering of the vertices of $B_e$. For $\eta \in \Xi$, we define $\eta_i$ by

$$\eta_i(x) = \begin{cases} 0 & \text{if } x \in \{v_1, v_2, \dots, v_i\}, \\ \eta(x) & \text{otherwise.} \end{cases} \tag{3.18}$$

Let $n$ be a positive integer, write $A = A_n$, and let $(\omega, \eta) \in \Omega \times \Xi$. Suppose for the moment that

$$m + 1 \le \|z\| \le n - m - 1, \tag{3.19}$$

and let

$$K_e = \min\big\{i : \text{ some vertex of } B_e \text{ is } (+)\text{pivotal for } A \text{ in the configuration } (\omega, \eta_i)\big\},$$

with the convention that the minimum of the empty set is $\infty$.

The configurations $(\omega, \eta_i)$ are obtained from $(\omega, \eta)$ by altering a bounded number of variables $\eta(x)$. Also, if $K_e < \infty$, then in at least one of the configurations $(\omega, \eta_i)$, $0 \le i \le |B_e|$, there exist one or more $(+)$pivotal vertices. Therefore there exists a function $\delta_1$ such that

(3.20)

$$\begin{aligned} P_{p,s}(K_e < \infty) &\le \sum_{i=0}^{|B_e|} \sum_{x \in B_e} P_{p,s}\big(\{(\omega, \eta) : x \text{ is } (+)\text{pivotal for } A \text{ in } (\omega, \eta_i)\}\big) \\ &\le \delta_1(p, s)\big(1 + |B_e|\big)^2 P_{p,s}(\Pi_e \ge 1), \end{aligned}$$

where $\Pi_e$ is the number of $(+)$pivotal vertices for $A$ lying in $B_e$; this may be compared with the final step in the proof of Lemma (3.5).

We consider next the case $K_e = \infty$. Let $(\omega, \eta) \in \Omega \times \Xi$ be such that $e$ is $(+)$pivotal for $A$, $\omega(e) = 1$, and $K_e = \infty$. Let $\eta'$ be given by

$$\eta'(x) = \begin{cases} 0 & \text{if } x \in B_e, \\ \eta(x) & \text{otherwise.} \end{cases}$$

Since $K_e = \infty$, we have that $e$ is $(+)$pivotal for $A$ in $(\omega, \eta')$. Using (3.19) and the fact that $\omega(e) = 1$, we observe that there exists an enhanced path $x_0 = 0,\ f_0, x_1,\ f_1, \dots, x_t$ with $x_t \in \partial B(n)$ which utilizes the edge $e$, and we set

$$r = \min\big\{i : x_i \in z + B(m)\big\}, \quad s = \max\big\{i : x_i \in z + B(m)\big\},$$

the first and last vertices thereof lying in $z + B(m)$. Note that $1 \le r < s < t$. Let $\mathbb{E}_e$ be the set of lattice edges having at least one endvertex in $z + B(m)$. We propose to alter the values $\omega(f)$, $f \in \mathbb{E}_e$, in order to obtain a new configuration in which $z$ is $(+)$pivotal for $A$. We do this by 'pasting' the configuration $\widehat{\omega} = \widehat{\omega}_m(x_r - z, x_s - z)$ into $z + B(m)$. More specifically, we define $\omega'$ $(\in \Omega)$ by

$$\omega'(h) = \begin{cases} \widehat{\omega}(h - z) & \text{if } h \text{ has both endvertices in } B_e, \\ 1 & \text{if } h = f_{r-1},\ f_s, \\ 0 & \text{for other edges } h \text{ of } \mathbb{E}_e, \\ \omega(h) & \text{if } h \notin \mathbb{E}_e. \end{cases} \tag{3.21}$$

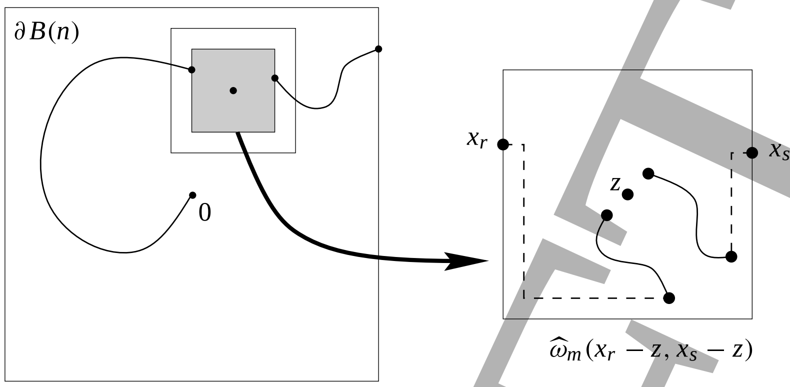


Figure 3.10. If $K_e = \infty$, one may alter the states of a bounded number of edges and vertices in order to obtain a configuration $\omega'$ in which $z$ is $(+)$pivotal for $A$.

The configuration $\omega'$ is illustrated in Figure 3.10.

It may be seen from the definition of $\widehat{\omega}$ that, in $(\omega', \eta')$, the vertex $z$ is $(+)$pivotal for $A$. Since $(\omega', \eta')$ has been obtained from $(\omega, \eta)$ by changing only a bounded number of variables $\omega(h)$, $\eta(x)$, there exists $\delta_2$ such that

$$
\begin{aligned}
(3.22)\qquad P_{p,s}\big(e \text{ is } (+)\text{pivotal}, e \text{ is open}, K_e = \infty\big) &\le \delta_2(p,s) P_{p,s}\big(z \text{ is } (+)\text{pivotal for } A\big)\\
&\le \delta_2(p,s) P_{p,s}(\Pi_e \ge 1).
\end{aligned}
$$

Adding (3.20) and (3.22), and remembering that the events $\{e \text{ is } (+)\text{pivotal for } A\}$ and $\{e \text{ is open}\}$ are independent, we conclude that

$$
(3.23)\qquad P_{p,s}\big(e \text{ is } (+)\text{pivotal for } A\big) \le \delta_3(p,s) P_{p,s}(\Pi_e \ge 1)
$$

for some $\delta_3$.

We now relax assumption (3.19). Suppose first that $z \in B(m)$ and that $e$ is $(+)$pivotal for $A$. Instead of working in the box $B_e = z + B(m)$, we work instead within the larger box $B(2m+1)$. If the quantity corresponding to $K_e$ is finite, then the above argument may be applied directly. If it is infinite, then we alter the configuration within $B(2m+R+1)$ in such a way as to arrange for the vertex $(m+1, 0, 0, \dots, 0)$ to become $(+)$pivotal for $A$. This leads as before to an inequality of the form of (3.23) with $\delta_3$ replaced by some $\delta_4$ and with $\Pi_e$ replaced by the number of $(+)$pivotal vertices inside the box $B(2m+R+1)$.

A similar construction is valid if $\|z\| \ge n - m$, although we note the added complication that there may exist $(+)$pivotal vertices which lie outside $B(n)$, but necessarily within distance $R$ of $B(n)$.

In conclusion, there exists $\delta_4$ such that, for all $e = \langle z, z+u\rangle \in \mathbb{E}^d$,

$$
(3.24)\qquad P_{p,s}\big(e \text{ is } (+)\text{pivotal for } A\big) \le \delta_4(p,s) P_{p,s}(\Pi'_e \ge 1)
$$

where $\Pi'_e$ is the number of pivotal vertices within $z + B(2m + R + 1)$.

Summing (3.24) over all $e \in \mathbb{E}^d$, we deduce via (3.17) that

$$\frac{\partial\theta_n}{\partial p} \le \delta_4(p,s)d\big|B(2m+R+1)\big|\frac{\partial\theta_n}{\partial s} = \nu(p,s)\frac{\partial\theta_n}{\partial s}, \tag{3.25}$$

just as in the second inequality of Lemma (3.5). We now argue as in the proof of (3.8)–(3.9). Let $\eta$ be positive and small, and choose $\gamma$ such that $\nu(p,s) \le \gamma^{-1}$ on $[\eta, 1-\eta]^2$, and let $\tan\psi = \gamma^{-1}$. If $(a,b) \in [2\eta, 1-2\eta]^2$ and

$$(a', b') = (a, b) + \eta(\cos\psi, -\sin\psi), \tag{3.26}$$

then, as in (3.9),

$$\theta(a', b') \le \theta(a, b). \tag{3.27}$$

Let $0 < b < 1$ and let $\eta$ $(> 0)$ be sufficiently small that

$$2\eta < b,\ p_c(\mathbb{L}^d) < 1 - 2\eta.$$

We may find $a$ such that

$$2\eta < a < p_c(\mathbb{L}^d) < a' < 1 - 2\eta$$

where $a'$ is given in (3.26). By (3.27),

$$\theta(a, b) \ge \theta(a', b') \ge \theta(a', 0) > 0,$$

whence $p_c^{\mathrm{enh}}(F, b) \le a$ as required. □

## 3.4 Bond and Site Critical Probabilities

For any connected graph $G$, it is the case that $p_c^{\mathrm{bond}}(G) \le p_c^{\mathrm{site}}(G)$, but when does strict inequality hold here? The answer depends on the choice of graph. For example, if $G$ is a tree, then it is easy to see that *equality* holds rather than *inequality*. On the other hand, it is reasonable to expect strict inequality to be valid for a range of graphs including all finite-dimensional lattices in two or more dimensions. We prove this in the special case of $\mathbb{L}^d$ with $d \ge 2$.

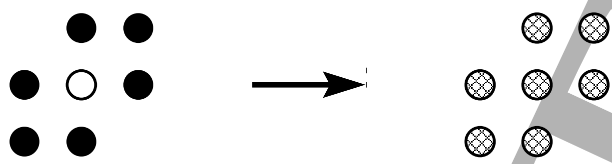

Figure 3.11. A representation of the enhancement described, when $d = 2$. Each copy of the configuration on the left is replaced, with probability $s$, by the configuration on the right. Filled circles indicate open vertices, and hatched circles denote enhanced vertices.

**(3.28) Theorem.** *Consider $\mathbb{L}^d$ with $d \geq 2$. We have that $p_c^{\text{bond}} < p_c^{\text{site}}$.*

**Proof.** We follow Grimmett and Stacey (1998). The basic approach is to use the enhancement technology expounded in the last section, but with some interesting differences. We shall construct a site percolation process on $\mathbb{L}^d$, and shall define an enhancement thereof which is dominated by bond percolation.

The sample space appropriate for site percolation is $\Xi = \{0, 1\}^{\mathbb{Z}^d}$. We interpret the vector $\xi \in \Xi$ as a realization of site percolation on $\mathbb{L}^d$. At each vertex $x$, we shall consider making an enhancement with probability $s$, and to this end we provide ourselves with an 'enhancement realization' $\eta \in \Xi$. As before, we interpret the value $\eta(x) = 1$ as meaning that the enhancement at the vertex $x$ is activated. The pair $(\xi, \eta)$ takes values in the sample space $\Xi \times \Xi$, and we endow this space with the product probability measure $P_{p,s} = P_p \times P_s$.

Let $u_1, u_2, \dots, u_d$ denote the unit vectors of $\mathbb{R}^d$, that is, $u_1 = (1, 0, 0, \dots, 0)$, $u_2 = (0, 1, 0, 0, \dots, 0)$, and so on. Given a vertex $x \in \mathbb{Z}^d$ we define disjoint sets of vertices close to $x$ as follows:

$$A_x = \left\{x + u_i,\ x + u_i + u_j : 1 \leq i < j \leq d\right\},$$
$$B_x = \left\{x - u_i,\ x - u_i - u_j : 1 \leq i < j \leq d\right\}.$$

We say that a vertex $x$ is a *qualifying vertex* (for $\xi$) if $\xi(x) = 0$ and $\xi(y) = 1$ for all $y \in A_x \cup B_x$. For $(\xi, \eta) \in \Xi \times \Xi$, the enhanced configuration $\zeta = \zeta(\xi, \eta)$ ($\in \Xi$) is defined by: $\zeta(x) = 1$ if and only if either

(i) $\xi(x) = 1$, or

(ii) $x$ is a qualifying vertex for $\xi$, and $\eta(x) = 1$.

We call a vertex $x$ *open* if $\xi(x) = 1$, and *enhanced* if $\zeta(x) = 1$. See Figure 3.11 for a sketch of the above enhancement in action.

We shall refer to $\zeta$ (or the law it induces on $\Xi$) as *enhanced site percolation with parameters $p$ and $s$*, and we write

$$\theta^{\text{enh}}(p, s) = P_{p,s}\big(0 \text{ lies in an infinite enhanced path}\big)$$

for the percolation probability of the enhanced configuration; cf. (3.14).

**(3.29) Lemma.** *We have that* $\theta^{\text{enh}}(p, p^2) \le \theta^{\text{bond}}(p)$.

Theorem (3.28) follows easily from this lemma, as follows. We note that Theorem (3.16) is not directly applicable in this setting, since it was concerned with enhancements of bond percolation rather than of site percolation. However, it is straightforward to adapt the theorem to the current setting, and it may be seen that the enhancement described above is essential in the sense of site percolation; see Paragraph E following the statement of Theorem (3.16). Let $s$ satisfy $\sqrt{s} = \frac{1}{2} p_{\text{c}}^{\text{site}}$. It follows from the appropriate reworking of Theorem (3.16) that there exists $\pi(s)$ satisfying $\pi(s) < p_{\text{c}}^{\text{site}}$ such that $\theta^{\text{enh}}(p, s) > 0$ for all $p > \pi(s)$. Let $p$ satisfy

$$\max\{\pi(s), \sqrt{s}\} < p < p_{\text{c}}^{\text{site}}.$$

Since $p^2 > s$, we have that $\theta^{\text{enh}}(p, p^2) \ge \theta^{\text{enh}}(p, s) > 0$. Therefore, by Lemma (3.29), $\theta^{\text{bond}}(p) > 0$, whence $p_{\text{c}}^{\text{site}} > p \ge p_{\text{c}}^{\text{bond}}$ as required.

**Proof of Lemma (3.29).** We shall employ a coupling of bond and site percolation which is essentially that used for the second inequality of (1.34). Let $X = (X_e : e \in \mathbb{E}^d)$ be a realization of bond percolation on $\mathbb{L}^d$. Let $Z = (Z_x : x \in \mathbb{Z}^d)$ be a collection of independent Bernoulli random variables, independent of the $X_e$, having mean $p$ also. In the first stage of this proof, we construct from these two families a new collection $Y = (Y_x : x \in \mathbb{Z}^d)$ of random variables, which constitutes a site percolation process with density $p$. This last process will have the property that, for $x, y \in \mathbb{Z}^d$, if $y$ cannot be reached from $x$ in the bond process $X$, then neither can $y$ be reached from $x$ in the site process $Y$; this will show that $\theta^{\text{site}}(p) \le \theta^{\text{bond}}(p)$.

Let $e_1, e_2, \dots$ be an enumeration of the edges of $\mathbb{Z}^d$ and let $x_1, x_2, \dots$ be an enumeration of its vertices; we take $x_1 = 0$, the origin. We wish to define the $Y_x$ in terms of the $X_e$ and the $Z_y$, and we shall do so by a recursion, described next. Suppose at some stage that we have defined the set $(Y_x : x \in S)$, where $S$ is a proper subset of $\mathbb{Z}^d$. (At the start we take $S = \varnothing$.) For $x \in S$, we say that $x$ is 'currently open' if $Y_x = 1$ and 'currently closed' if $Y_x = 0$. Let $T$ be the set of vertices not belonging to $S$ which are adjacent to some currently open vertex. If $T = \varnothing$, then let $y$ be the first vertex (in the above enumeration) not lying in $S$, and set $Y_y = Z_y$. If $T \ne \varnothing$, we let $y$ be the first vertex in $T$, and we let $y'$ be the first currently open vertex adjacent to it; we then set $Y_y = X_{\langle y, y' \rangle}$, where as usual $\langle u, v \rangle$ denotes the edge joining two neighbours $u, v$. Repeating this procedure will eventually exhaust all vertices $x \in \mathbb{Z}^d$, and assign values to all the variables $Y_x$.

This algorithm begins at the origin 0, and builds up a (possibly infinite) open cluster together with a neighbour set of closed vertices. When the cluster at 0 is complete, another vertex is selected as a new starting point, and the process is iterated. Note that this recursion is transfinite, since infinitely many steps are needed in order to build up any infinite cluster.

We now make two observations about the variables $Y_x$. First, for each vertex $x$, the probability that $Y_x = 1$, conditional on any information about the values of those $Y_y$ determined prior to the definition of $Y_x$, is equal to $p$. Based upon this observation one may prove without great difficulty that the random variables $(Y_x : x \in \mathbb{Z}^d)$ are independent with mean $p$, which is to say that they form a site percolation process on $\mathbb{Z}^d$.

Secondly, if there exists a path of open vertices between two points, then there exists a (possibly longer) path of open bonds. Therefore we have succeeded in coupling a bond and a site process with the required domination property.

We shall now adapt this construction in order to obtain a suitable coupling of bond percolation with the enhanced site percolation process obtained from the $Y_x$. Here is the main idea. Suppose that $x$ is a qualifying vertex for the realization $Y$. Then $Y_x = 0$, and $Y_y = 1$ for all $y \in A_x \cup B_x$. Note that all the vertices of $A_x$ (respectively $B_x$) must lie in the same site percolation cluster $C_1 = C_1(x)$ (respectively $C_2 = C_2(x)$). If $C_1 = C_2$, then the enhancing of $x$ makes no difference to the connectivity properties of the graph except at $x$. If $C_1 \neq C_2$, then enhancing $x$ effectively joins $C_1$ and $C_2$ together. Since $Y_x = 0$, it is the case that at most one edge $e$ incident with $x$ was examined (in the sense that the value of $X_e$ was considered) in the determination of the $Y_u$. Therefore, there exists at least one unexamined edge joining $x$ to $A_x$; let the first such edge in our enumeration be $e = e(x)$. Likewise, there exists a first unexamined edge, $f = f(x)$ say, joining $x$ and $B_x$. We adopt the following rule: we declare $x$ to be enhanced if and only if $X_e = X_f = 1$. This has the effect of adding $x$ into the enhanced configuration with probability $p^2$. Acting thus for all qualifying vertices $x$ yields an enhanced site percolation; the independence of the enhancement at different qualifying vertices follows from the fact that the sequence of all $e(x)$ and $f(x)$ contains no repetitions. Furthermore, the above enhancement cannot join any two vertices which are not already joined by an open path in the bond model: enhancing $x$ has the effect of connecting $x$ to the clusters $C_1(x)$ and $C_2(x)$ and to no others, and this enhancement of $x$ occurs only in situations where $x$ is already joined to both of these clusters in the bond process $X$.

It is fairly straightforward to present a formal description of the informal account above. In order to obtain the appropriate enhancement, we require a family $(H_x : x \in \mathbb{Z}^d)$ of independent Bernoulli random variables, having parameter $p^2$ and independent of the vector $Y$. We only require the $H_x$ for qualifying vertices $x$, and we may simply set $H_x = X_{e(x)}X_{f(x)}$, where $e(x)$ and $f(x)$ are given as above.

We have now given a coupling of bond percolation and an enhanced site percolation with the property that any two vertices which are in the same cluster of the enhanced site process are also in the same cluster of the bond process. It follows that, if the origin lies in an infinite enhanced path, then the cluster containing the origin in the bond process is infinite also. The required inequality follows. □

## 3.5 Notes

**Section 3.1.** We omit a detailed history of the results of this section, of which a discussion may be found in Hughes (1996). Kesten (1980a, 1982) proved that $p_c = \frac{1}{2}$ for bond percolation on $\mathbb{L}^2$, and Wierman (1981) adapted his proof in order to calculate $p_c$ for the hexagonal and triangular lattices. These rigorous arguments confirmed the proposals of Sykes and Essam (1963, 1964), who discussed the notion of a matching pair of graphs. The exact calculation of $p_c$(bow-tie) appeared in Wierman (1984a).

Certain rigorous numerical inequalities have been proved for two-dimensional percolation by Wierman (1990, 1995). The rigorous derivation of the series expansion (3.2) was presented by Hara and Slade (1995), in response to physical arguments which appeared earlier in the physics literature.

**Sections 3.2 and 3.3.** The first systematic approach to strict inequalities for ordered pairs of lattices is due to Menshikov (1987a, d, e), although there existed already some special results in the literature. The discussion and technology of Sections 3.2 and 3.3 draws heavily on Aizenman and Grimmett (1991); see also Grimmett (1997).

Theorem (3.16) may be adapted to enhancements of site percolation (see the discussion following the statement of the theorem). The assumption that enhancements take place at *all* vertices $x$ may be relaxed; see Aizenman and Grimmett (1991).

The problem of entanglements appeared first in Kantor and Hassold (1988), who reported certain numerical conclusions. The existence of an entanglement transition different from that of percolation was proved by Aizenman and Grimmett (1991); the strict positivity of the entanglement critical point was proved by Holroyd (1998b). The entanglement transition has been studied more systematically by Holroyd (1998b) and Grimmett and Holroyd (1998). There are topological difficulties in deciding on the 'correct' definition of critical point, and in proving that the critical point differs from zero. Certain related issues arise in the study of so called 'rigidity percolation', in which one studies the existence of infinite rigid components of the open subgraph of a lattice; see Jacobs and Thorpe (1995, 1996) and Holroyd (1998a). Further accounts of entanglement and rigidity may be found in Sections 12.5 and 12.6.

The 'augmented percolation' question posed after Theorem (3.16) was discussed by Chayes, Chayes, and Newman (1988) in the context of invasion percolation on the triangular lattice and on the covering lattice of the square lattice. It was answered by Aizenman and Grimmett (1991).

**Section 3.4.** Theorem (3.28) is taken from Grimmett and Stacey (1998), where a general theorem of this sort is presented. Earlier work on strict inequalities between bond and site critical probabilities in two dimensions may be found in Higuchi (1982), Kesten (1982), and Tóth (1985). Corresponding results for Ising, Potts, and random-cluster models have been studied by Aizenman and Grimmett (1991), Bezuidenhout, Grimmett, and Kesten (1993), and by Grimmett (1993, 1994a, 1995a, 1999c).