

The 1-2 model

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ABSTRACT. The current paper is a short review of rigorous results for the 1-2 model. The 1-2 model on the hexagonal lattice is a model of statistical mechanics in which each vertex is constrained to have degree either 1 or 2. It was proposed in a study by Schwartz and Bruck of constrained coding systems, and is strongly connected to the dimer model on a decoration of the lattice, and to an enhanced Ising model and an associated polygon model on the graph derived from the hexagonal lattice by adding a further vertex in the middle of each edge.

The general 1-2 model possesses three parameters a , b , c . The fundamental technique is to represent probabilities of interest as ratios of counts of dimer coverings of certain associated graphs, and to apply the Pfaffian method of Kasteleyn, Fisher, and Temperley.

Of special interest is the existence (or not) of phase transitions. It turns out that all clusters of the infinite-volume limit are almost surely finite. On the other hand, the existence (with strictly positive probability) of infinite ‘homogeneous’ clusters, containing vertices of given type, depends on the values of the parameters.

A further type of phase transition emerges in the study of the two-edge correlation function, and in this case the critical surface may be found explicitly. For instance, when $a \geq b \geq c > 0$, the surface given by $\sqrt{a} = \sqrt{b} + \sqrt{c}$ is critical.

1. Origin of the 1-2 model

The 1-2 model originated in the work of computer scientists Schwartz and Bruck [23] on constrained coding systems. They studied an array of variables on the hexagonal lattice \mathbb{H} subject to the ‘not all equal’ constraint. Of particular interest to them was the asymptotic behaviour of the number of acceptable configurations on large bounded regions \mathbb{H}_n of the lattice, in the ‘thermodynamic limit’ as $\mathbb{H}_n \uparrow \mathbb{H}$. Using the method of so-called ‘holographic reduction’, they were able to map their counting problem to one of counting the number of perfect matchings (or ‘dimer coverings’) on a certain graph derived from the hexagonal lattice. This last problem may be solved using the Pfaffian representation of Kasteleyn [15], Fisher [5], and Temperley and Fisher [24].

When rephrased in the language of statistical mechanics, the work of Schwartz and Bruck amounts to the calculation of the partition function of the following

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1-2 model of probability theory and mathematical physics. Let $\mathbb{H} = (\mathbb{V}, \mathbb{E})$ be the hexagonal lattice of Figure 1.1, and let $\Sigma = \{-1, 1\}^{\mathbb{E}}$ be the set of configurations of absent/present edges, where the local state -1 (respectively, 1) means *absent* (respectively, *present*). The sample space is the subset of Σ containing all $\sigma \in \Sigma$ such that: every vertex of \mathbb{H} is incident to either one or two present edges. Thus, a configuration comprises disjoint paths and cycles of present edges.

Let \mathbb{H}_n be an $n \times n$ subgraph of \mathbb{H} with periodic boundary conditions, and let μ_n be the uniform probability measure on the set of 1-2 configurations on \mathbb{H}_n . We ask for properties of μ_n in the limit as $n \rightarrow \infty$. In particular, does the limit measure $\mu_\infty := \lim_{n \rightarrow \infty} \mu_n$ exist, and, if so, what can be said about the long-range correlations of edge-states under μ_∞ ? It turns out that the connection to dimers may be exploited to answer such questions.

The above system is a lattice model whose partition function can be computed by the calculation of certain determinants using the holographic algorithm of Valiant [25]. By introducing an invertible 2×2 matrix on edges of a graph and conducting a base change, the partition function of a general vertex model on a graph G is transformed into the partition function of perfect matchings on a certain decorated version of G . Valiant's original holographic algorithm can be generalized by assigning different bases to different edges (see [16]), and the ensuing algorithm can be used to compute partition functions of a larger class of vertex models in polynomial time.

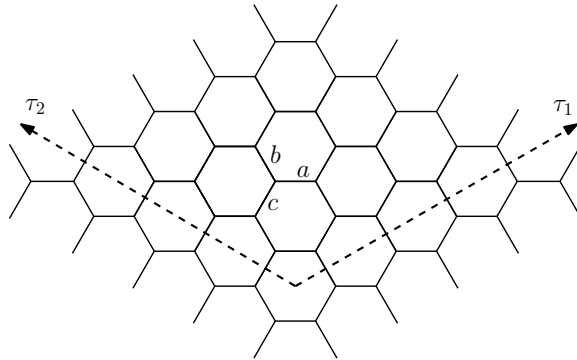


FIGURE 1.1. The graph \mathbb{H}_n is an $n \times n$ lozenge wrapped onto a torus. A horizontal (respectively, NW/SE, NE/SW) edge is said to be of type a (respectively, type b , c).

The holographic algorithm seems, however, not to be the most efficient way to solve the 1-2 model. In particular, the correspondence between the 1-2 partition function and the dimer partition function on a corresponding Fisher graph, via the base change method, is not measure-preserving; thus, the computation of local statistics and related probabilities becomes complicated, even if possible. An alternative measure-preserving correspondence was introduced in [18], and this permits a number of representations in closed form of probabilities associated with the 1-2 model. This method, and some of its consequences, will be described in the current review.

Certain properties of the underlying hexagonal lattice are utilized heavily in this work, such as trivalence, planarity, and support of a \mathbb{Z}^2 action. It may be possible

to extend the results summarized here to certain other graphs with such properties, including the Archimedean $(3, 12^2)$ and $(4, 8^2)$ lattices, and also to certain periodic models.

The formal definition of the 1-2 model is presented in Section 2. The model has strong connections to the dimer and Ising models as well as to a certain polygon model, and these connections are laid out in Section 3. Two approaches to the issue of phase transition are outlined in Section 4, using the geometry and the correlation structure, respectively, and an exact formula for the critical surface is given in the second case.

2. Definition of the 1-2 model

Whereas the 1-2 model of [23] is uniform in that there is only one parameter, we present here the more general three-parameter model of [16].

Let $n \geq 1$, and let τ_1, τ_2 be the two shifts of \mathbb{H} as in Figure 1.1. The pair (τ_1, τ_2) generates a \mathbb{Z}^2 action on \mathbb{H} , and we write $\mathbb{H}_n = (V_n, E_n)$ for the (toroidal) quotient graph of \mathbb{H} under the subgroup of \mathbb{Z}^2 generated by the powers τ_1^n and τ_2^n . The configuration space Σ_n is the set of all $\sigma \in \{-1, 1\}^{E_n}$ such that every $v \in V_n$ is incident to either 1 or 2 edges e with $\sigma(e) = 1$. Note that $\sigma \in \Sigma_n$ if and only if $-\sigma \in \Sigma_n$. It is sometimes convenient to work with the vector σ' given by $\sigma'(e) = \frac{1}{2}(1 + \sigma(e))$.

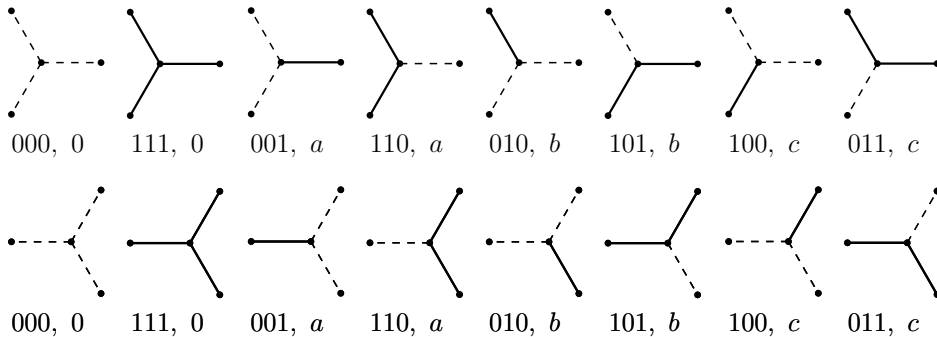


FIGURE 2.1. Each vertex has eight possible local configurations, six of which have nontrivial weights, a, b , or c as above. A vertex v is said to be of *type* s in configuration σ if $\sigma|_v$ has weight $s \in \{a, b, c\}$. Whereas the type of an edge is *deterministic* (see Figure 1.1), the type of a vertex is *random*.

A vertex $v \in V_n$ is incident to three edges e_1, e_2, e_3 of \mathbb{H}_n , which are in the respective orientations: horizontal, NW/SE, and NE/SW. Let $\sigma \in \Sigma_n$, and let the *signature* at v be the triple $\sigma|_v := \sigma'(e_3)\sigma'(e_2)\sigma'(e_1) \in \{0, 1\}^3$ considered as a word with three letters in the alphabet with two letters. Let $a, b, c \in [0, \infty)$ be such that $(a, b, c) \neq (0, 0, 0)$. To the signature $\sigma|_v$ is allocated the weight $w(\sigma|_v) \in \{0, a, b, c\}$ given in Figure 2.1, and the weight function $w : \Sigma_n \rightarrow [0, \infty)$ is defined by

$$(2.1) \quad w(\sigma) = \prod_{v \in V_n} w(\sigma|_v).$$

This gives rise to the probability measure μ_n given by

$$(2.2) \quad \mu_n(\sigma) = \frac{1}{Z_n} w(\sigma),$$

where

$$(2.3) \quad Z_n = Z_n(a, b, c) = \sum_{\sigma \in \Sigma_n} w(\sigma)$$

is the *partition function*. A sample drawn (approximately) from μ_{10} is depicted in Figure 2.2.

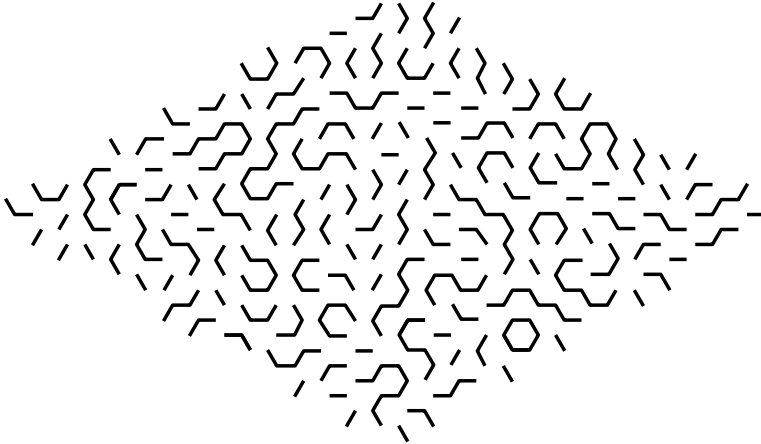


FIGURE 2.2. A realization of the (uniform) 1-2 model on \mathbb{H}_{10} with $a = b = c = 1$, drawn by MCMC methods. Each vertex has degree either 1 or 2, and therefore all components are either paths or cycles.

It turns out that the weak limit of the sequence $(\mu_n : n \geq 1)$ exists. However, no simple correlation inequality is known, and the proof of existence of the limit follows a different route using the relationship to dimer configurations outlined in Section 3.1.

THEOREM 2.1 ([10, Thm 6.2]). *The weak limit*

$$\mu_\infty := \lim_{n \rightarrow \infty} \mu_n$$

exists and is translation invariant.

The infinite-volume limit μ_∞ is of course a Gibbs state in the sense that it satisfies the relevant Dobrushin–Lanford–Ruelle (DLR) condition (see [7, Sect. 4.4]). On the other hand, the structure of the space of such Gibbs measures is unknown. Neither is it known for which parameter-values μ_∞ is ergodic (it is not ergodic under the conditions of [18, Thm 4.9] and Theorem 4.3(b) of the current paper).

REMARK 2.2. *For edges e, f of \mathbb{H} and for sufficiently large n , we write $\langle \sigma_e \sigma_f \rangle_n$ for the two-edge correlation function of the measure μ_n . By Theorem 2.1, the limit*

$$\langle \sigma_e \sigma_f \rangle = \lim_{n \rightarrow \infty} \langle \sigma_e \sigma_f \rangle_n$$

exists. In Section 4.3, we shall consider the limit $\lim_{|e-f| \rightarrow \infty} \langle \sigma_e \sigma_f \rangle$ as an order parameter that is indicative of phase transition.

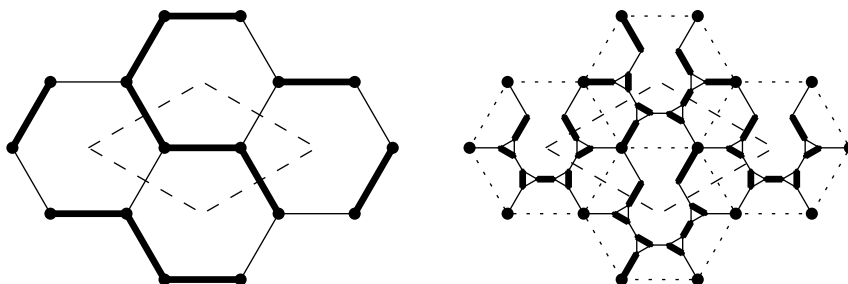


FIGURE 2.3. Part of the hexagonal lattice \mathbb{H} is drawn on the left, together with a 1-2 configuration. The graph \mathbb{H}_Δ on the right is obtained by replacing each face of \mathbb{H} by a certain ‘gadget’. The left-hand 1-2 configuration gives rise to a dimer configuration on the decorated graph on the right, as described in the text. The fundamental domain is outlined as a lozenge, and expanded in Figure 3.1.

3. The dimer, Ising, and polygon models

The relationship between the 1-2 model and the dimer model is pivotal to the study of the former. Dimers are relevant also to the study of Ising models in two dimensions (see, for example, [17]), and the Ising model gives rise in turn to a ‘high temperature’ polygon model (see for example, [1, p. 75] and [9, 21, 26]). Therefore, the 1-2 model is connected firmly to the Ising and polygon models. These connections play roles in the theory of the 1-2 model, and are summarized in this section.

3.1. The dimer model. Let $\mathbb{H} = (\mathbb{V}, \mathbb{E})$ be the hexagonal lattice, and let $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$ be the ‘decorated graph’ drawn on the right side of Figure 2.3. The graph in the figure is obtained by replacing each face F of \mathbb{H} by a certain ‘gadget’ comprising a path which is joined to the vertices of F in the manner drawn in the figure.

A 1-2 configuration of the left side of Figure 2.3 gives rise to a dimer configuration on the decorated graph on the right, as follows. Note first that $\mathbb{V} \subseteq V_\Delta$. Each $v \in \mathbb{V}$ has three incident edges in \mathbb{H}_Δ , and these edges in \mathbb{H}_Δ may be regarded as the *bisector edges* of the three angles in \mathbb{H} at v .

Let $\sigma \in \Sigma$. An edge $e \in E_\Delta$, incident to a vertex $v \in \mathbb{V}$, is designated *present* if and only if the two sides of the corresponding angle of \mathbb{H} have the same states, that is, either both are present or both are absent. Once we have determined the states of the bisector edges of \mathbb{H}_Δ , there is a unique extension to a dimer configuration on \mathbb{H}_Δ . See Figure 2.3. Note that the two 1-2 configurations $\sigma, -\sigma \in \Sigma$ give rise to the same dimer configuration, and thus the above correspondence is two-to-one.

Consider now the toroidal graph \mathbb{H}_n and the corresponding decorated graph $\mathbb{H}_{n,\Delta}$. The edges of $\mathbb{H}_{n,\Delta}$ are weighted, with the edge $e = \langle i, j \rangle$ having weight

$$(3.1) \quad w_{i,j} = \begin{cases} a & \text{if } e \text{ is a horizontal bisector edge,} \\ b & \text{if } e \text{ is a NW/SE bisector edge,} \\ c & \text{if } e \text{ is a NE/SW bisector edge,} \\ 1 & \text{otherwise.} \end{cases}$$

The weight of a dimer configuration is defined as the product of the weights of the edges that are present, and this gives rise (as in (2.2)) to a probability measure $\delta_{n,\Delta}$ on dimer configurations. Now, $\delta_{n,\Delta}$ -probabilities may be represented as weighted counts of dimer configurations, and such quantities on planar graphs may be computed by the Pfaffian method of Kasteleyn [15], Fisher [5], and Temperley and Fisher [24]. This leads to the following limit theorem.

THEOREM 3.1 ([18, Prop. 3.3]). *Let $a, b, c > 0$. The limit measure*

$$\delta_\Delta := \lim_{n \rightarrow \infty} \delta_{n,\Delta}$$

exists and is translation-invariant and ergodic.

Theorem 2.1 follows by Theorem 3.1 and the above correspondence between 1-2 model configurations and dimer configurations. Note that the weak limit for 1-2 measures need not be ergodic.

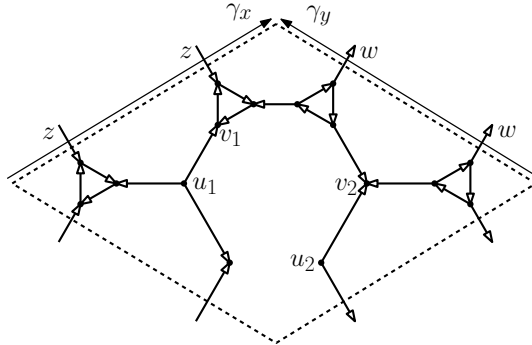


FIGURE 3.1. A single fundamental domain of the decorated graph $\mathbb{H}_{n,\Delta}$ obtained from the central lozenge of Figure 2.3. This illustrates the relationship between the fundamental domain and the original hexagonal lattice \mathbb{H} . Note the homology cycles γ_x, γ_y of the torus.

A $\delta_{n,\Delta}$ -probability may be expressed in terms of weighted counts of dimers, and these are studied via the Pfaffian representation. As explained in [10] and the references therein, the asymptotics (as $n \rightarrow \infty$) of such Pfaffians depend on the so-called *characteristic polynomial* of the model. We do not define the characteristic polynomial here beyond saying that it is the determinant $P(z, w)$ of the weighted adjacency matrix of the fundamental domain of \mathbb{H}_Δ , oriented in a ‘clockwise odd’ manner, and illustrated in Figure 3.1. It is a function of the parameters a, b, c , and

of two complex variables z, w . It is shown in [18] that

$$P(z, w) = a^4 + b^4 + c^4 + 6a^2b^2 + 6a^2c^2 + 6b^2c^2 - 2ab \left(z + \frac{1}{z} \right) (a^2 + b^2 - c^2) - 2ac \left(w + \frac{1}{w} \right) (a^2 + c^2 - b^2) - 2bc \left(\frac{z}{w} + \frac{w}{z} \right) (b^2 + c^2 - a^2).$$

The *spectral curve* is the zero locus of the characteristic polynomial, that is, the set of roots of the equation $P(z, w) = 0$. As explained in [18], it is important to understand the intersection of the spectral curve with the unit torus

$$\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}.$$

It turns out that the intersection is either empty or is a single real point $(1, 1)$. Moreover, when $P(1, 1) = 0$, the zero $(1, 1)$ has multiplicity 2. Evidently,

$$(3.2) \quad P(1, 1) = (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac)^2,$$

and therefore the spectral curve intersects \mathbb{T}^2 if and only if

$$(3.3) \quad \sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0.$$

We shall return to this equation in the study of phase transition in Section 4.3.

3.2. The half-edge graph. When considering correlations, it will be convenient to work on a graph derived from the hexagonal lattice by replacing each edge by two half-edges. Let $A\mathbb{H}_n = (AV_n, AE_n)$ be the graph derived from $\mathbb{H}_n = (V_n, E_n)$ by adding a vertex at the midpoint of each edge in E_n . Let $ME_n = \{Me : e \in E_n\}$ be the set of such midpoints, and $AV_n = V_n \cup ME_n$. The edges AE_n are the half-edges of E_n , each being of the form (v, Me) for some $v \in V_n$ and incident edge $e \in E_n$.

The 1-2 model on \mathbb{H}_n can be viewed as a spin-model on the set ME_n of midpoints, as we explain next.

3.3. The Ising model. It turns out that the 1-2 model is the marginal of a certain Ising-type model on the half-edge graph $A\mathbb{H}_n$ of Section 3.2, that is reminiscent of the Edwards–Sokal coupling of the Potts and random-cluster measures (see [7, Sect. 1.4]). It is constructed via a weight function on configuration space, using weights that are permitted in general to be \mathbb{C} -valued.

Let $\Sigma^e = \{-1, +1\}^{ME_n}$ and $\Sigma^v = \{-1, +1\}^{V_n}$. An edge $e \in E_n$ is identified with the element of ME_n at its centre. A spin-vector is a pair $(\sigma^e, \sigma^v) \in \Sigma^e \times \Sigma^v$ with $\sigma^e = (\sigma_{v,s} : v \in V_n, s = a, b, c)$ and $\sigma^v = (\sigma_v : v \in V_n)$, where $\sigma_{v,a}, \sigma_{v,b}, \sigma_{v,c}$ denote the spins on midpoints of the corresponding edges of given types incident to $v \in V_n$ (see Figure 1.1). We allocate the (possibly complex) weights

$$(3.4) \quad w(\sigma^v, \sigma^e) = \prod_{v \in V_n} (1 + \epsilon_a \sigma_v \sigma_{v,a})(1 + \epsilon_b \sigma_v \sigma_{v,b})(1 + \epsilon_c \sigma_v \sigma_{v,c}),$$

where $\epsilon_a, \epsilon_b, \epsilon_c \in \mathbb{C}$ are constants associated with horizontal, NW/SE, and NE/SW edges, respectively. In (3.4), each factor $1 + \epsilon_s \sigma_v \sigma_{v,s}$ ($s = a, b, c$) corresponds to a half-edge of \mathbb{H}_n .

When considering the relationship to the 1-2 model, it will be convenient to choose the constants $\epsilon_s, s = a, b, c$, as follows. Let $a, b, c > 0$, and

$$(3.5) \quad A = \frac{a - b - c}{a + b + c}, \quad B = \frac{b - a - c}{a + b + c}, \quad C = \frac{c - a - b}{a + b + c},$$

where we assume for simplicity that $ABC \neq 0$. The appropriate values of the ϵ_s are

$$(3.6) \quad \epsilon_a = \sqrt{\frac{BC}{A}}, \quad \epsilon_b = \sqrt{\frac{AC}{B}}, \quad \epsilon_c = \sqrt{\frac{AB}{C}}.$$

THEOREM 3.2 ([10, Sect. 4]).

(a) Marginal on V_n . For $\sigma^v \in \Sigma^v$, $w(\sigma^v) := \sum_{\sigma^e \in \Sigma^e} w(\sigma^v, \sigma^e)$ satisfies

$$(3.7) \quad w(\sigma^v) = 2^{|E_n|} \prod_{g=\langle u,v \rangle \in E_n} (1 + \epsilon_g^2 \sigma_u \sigma_v).$$

That is, the marginal weights on Σ^v are those of an Ising-type model on \mathbb{H}_n with (possibly complex) edge-interactions. Here, ϵ_g denotes the parameter associated with edge $g = \langle u, v \rangle$.

(b) Marginal on ME_n . Let $a, b, c > 0$ and assume the ϵ_s satisfy (3.5)–(3.6) where $ABC \neq 0$. For $\sigma^e \in \Sigma^e$, $w(\sigma^e) := \sum_{\sigma^v \in \Sigma^v} w(\sigma^v, \sigma^e)$ satisfies

$$w(\sigma^e) = \prod_{v \in V_n} (1 + A\sigma_{v,b}\sigma_{v,c} + B\sigma_{v,a}\sigma_{v,c} + C\sigma_{v,a}\sigma_{v,b}) \propto \mu_n(\sigma^e).$$

That is, the marginal weights on Σ^e are proportional to those of the 1-2 model.

(c) Two-edge correlation function. Let $e = \langle u, v \rangle, f = \langle x, y \rangle \in E_n$. Subject to the notation of part (b) above, the two-edge correlation function of Remark 2.2 satisfies

$$(3.8) \quad \langle \sigma_e \sigma_f \rangle_n = \sum_{\sigma^v \in \Sigma^v} D_{e,f}(\sigma^v) w(\sigma^v) \Big/ \sum_{\sigma^v \in \Sigma^v} w(\sigma^v),$$

where

$$D_{e,f}(\sigma^v) = \frac{\epsilon_e(\sigma_u + \sigma_v)\epsilon_f(\sigma_x + \sigma_y)}{(1 + \epsilon_e^2)(1 + \epsilon_f^2)}.$$

If the weights $w(\sigma^v)$ of (3.7) are real and non-negative (which they are not in general), the ratio on the right side of (3.8) may be interpreted as an expectation. The weights $w(\sigma^v)$ correspond to a ferromagnetic Ising model if and only the edge-weights of (3.7) satisfy $0 < \epsilon_g^2 < 1$. Unfortunately, this never occurs with the ϵ_g derived from the 1-2 model as in (3.5)–(3.6). If, however, one assumes that the a, b, c satisfy the acute angle condition

$$a^2 < b^2 + c^2, \quad b^2 < c^2 + a^2, \quad c^2 < a^2 + b^2,$$

then $-1 < \epsilon_g^2 < 0$, and the corresponding Ising model is antiferromagnetic. Since \mathbb{H} is bipartite, this process may be transformed into a ferromagnetic system by changing the sign of every other vertex (see [6, p. 17]), and this transformation greatly facilitates its analysis.

The above Ising model may be regarded as a special case of the eight-vertex model of Lin and Wu [20].

3.4. The hexagonal polygon model. Let $\mathbb{H}_n = (V_n, E_n)$ as before, and let $\Pi_n = \{0, 1\}^{E_n}$. The sample space of the polygon model on \mathbb{H}_n is the subset $\Pi_n^{\text{poly}} \subseteq \Pi_n$ containing all $\pi = (\pi(e) : e \in E_n) \in \Pi_n$ such that

$$(3.9) \quad \sum_{e \ni v} \pi(e) \quad \text{is either 0 or 2,} \quad v \in V_n.$$

Each $\pi \in \Pi_n^{\text{poly}}$ may be considered as a union of vertex-disjoint cycles of \mathbb{H}_n , together with isolated vertices. We identify $\pi \in \Pi_n$ with the set $\{e \in E_n : \pi(e) = 1\}$ of ‘open’ edges under π . Thus (3.9) requires that every vertex is incident to an even number of open edges.

Let $\epsilon_a, \epsilon_b, \epsilon_c \in \mathbb{C}$. To the configuration $\pi \in \Pi_n^{\text{poly}}$, we assign the (possibly complex) weight

$$(3.10) \quad w(\pi) = \epsilon_a^{2|\pi(a)|} \epsilon_b^{2|\pi(b)|} \epsilon_c^{2|\pi(c)|},$$

where $\pi(s)$ is the set of open s -type edges. The weight function w gives rise to the partition function

$$Z_n^{\text{poly}} = \sum_{\pi \in \Pi_n^{\text{poly}}} w(\pi).$$

Let $a, b, c > 0$. We now choose the constants ϵ_s , $s = a, b, c$, according to (3.5)–(3.6), where it is assumed that $ABC \neq 0$. The corresponding polygon model is related to the high-temperature expansion of the Ising-type model of Section 3.3 (see, for example, [8, eqn 5.1] and [9, Thm 1.7]).

In considering correlation functions, it is convenient to view the polygon model on the half-edge graph $A\mathbb{H}_n$ of Section 3.2. A polygon configuration on \mathbb{H}_n induces a polygon configuration on $A\mathbb{H}_n$, namely a subset of AE_n with the property that every vertex in AV_n has even degree. For an a -type edge $e \in E_n$, the two half-edges of e have weight ϵ_a each (and similarly for b - and c -type edges). The weight function w of (3.10) may now be expressed as

$$w(\pi) = \epsilon_a^{|\pi(a)|} \epsilon_b^{|\pi(b)|} \epsilon_c^{|\pi(c)|},$$

where $\pi(s)$ is the set of open half-edges of type s , as π ranges over polygon configurations on $A\mathbb{H}_n$.

Let $e, f \in ME_n$ be distinct midpoints of $A\mathbb{H}_n$, and let $\Pi_{e,f}$ be the subset of all $\pi \in \{0, 1\}^{AE_n}$ such that: (i) every $v \in AV_n$ with $v \neq e, f$ is incident to an even number of open half-edges, and (ii) the midpoints e and f are incident to exactly one open half-edge. Let

$$(3.11) \quad M_n(e, f) = \frac{Z_{n,e \leftrightarrow f}}{Z_n^{\text{poly}}},$$

where

$$(3.12) \quad Z_{n,e \leftrightarrow f} := \sum_{\pi \in \Pi_{e,f}} \epsilon_a^{|\pi(a)|} \epsilon_b^{|\pi(b)|} \epsilon_c^{|\pi(c)|}.$$

THEOREM 3.3 ([12]). *Subject to (3.6), the two-edge correlation function $\langle \sigma_e \sigma_f \rangle_n$ of the 1-2 model on \mathbb{H}_n satisfies $\langle \sigma_e \sigma_f \rangle_n = M_n(e, f)$.*

The polygon model with general parameters is studied in [12].

4. Phase transition

We discuss two forms of phase transition for the 1-2 model. Of these, the first concerns the existence (or not) of infinite ‘homogeneous’ clusters of \mathbb{H} containing vertices of the same type, and the second considers as order parameter the limiting two-edge correlation function. Thus, the first studies the geometry of the model, and the second its correlation structure. There may exist other forms of phase transition, as yet unstudied.

4.1. Occurrence of paths. Every connected component (‘cluster’) in a realization of the 1-2 model is either a self-avoiding path or a cycle (see Figure 2.2). It turns out that all such clusters are μ_∞ -a.s. finite, when $a, b, c > 0$. We formalize this statement in this subsection, and begin with an exact formula. Using the correspondence between 1-2 model configurations on \mathbb{H} and dimer configurations on \mathbb{H}_Δ , as described in Section 3.1, we have the following.

THEOREM 4.1 ([18, Thm 3.4]). *Let $a, b, c > 0$ and let μ_∞ be the limit 1-2 measure of Theorem 2.1. Let ℓ be a self-avoiding path of \mathbb{H}_n containing $l + 1$ edges, and write $E_\ell = \{e_k = \langle u_k, v_k \rangle : 1 \leq k \leq l\}$ for the set of bisector edges of \mathbb{H}_Δ encountered along ℓ , as in Figure 2.3. Then*

$$\mu_\infty(\ell \text{ is present}) = \frac{1}{2} \left(\prod_{k=1}^l w_{e_k} \right) |\text{Pf } K_\ell^{-1}|,$$

where w_e is the weight of the edge e in \mathbb{H}_Δ , K_ℓ^{-1} is the submatrix of the inverse of the weighted adjacency matrix of \mathbb{H}_Δ with rows and columns indexed by $u_1, v_1, \dots, u_l, v_l$, and $\text{Pf } M$ is the Pfaffian of the matrix M .

The mass transport principle, introduced in [3, 13], is a valuable tool in the study of interacting systems including percolation and self-avoiding walks on Cayley graphs, see [2, 11, 14]. It may also be applied to the 1-2 model, where it is used to prove the following.

THEOREM 4.2 ([18, Thm 2.4]). *If $a, b, c > 0$, we have that*

$$\mu_\infty(\text{there exists an infinite path}) = 0.$$

4.2. Existence of infinite homogeneous clusters. The concept of ‘phase transition’ hinges on the non-smoothness of some so-called ‘order parameter’. For the Ising model, one may take as order parameter the magnetization at the origin in the infinite-volume measure with + boundary conditions. This corresponds in the universe of percolation and the random-cluster model (see [7]) to studying whether or not there exists an infinite cluster. By Theorem 4.2, the 1-2 model possesses no infinite cluster for any values of $a, b, c > 0$. ‘Clusters’ may however be defined in another manner.

Let $\sigma \in \Sigma$. Each vertex of \mathbb{H} has a random type, given in Figure 2.1. For $s \in \{a, b, c\}$, a *type- s* cluster is a maximal connected subgraph of \mathbb{H} every vertex of which has type s . As illustrated in the figure, type s comes in two sub-types; for example, a type- a vertex has signature either 001 or 110. Thus, one may speak of a 001-cluster, etc. By examining the figure again, it is seen that a type- a cluster is either a 001-cluster or a 110-cluster, but may not contain vertices of both types (and similarly for types b and c). A *homogeneous cluster* of $\sigma \in \Sigma$ is a w -cluster

for some $w \in \{0, 1\}^3$, $w \neq 000, 111$. We concentrate now on the existence (or not) of an infinite homogeneous cluster.

THEOREM 4.3. *Let $a, b, c > 0$.*

- (a) ([19, Thm 1.1]) *Let $w \in \{0, 1\}^3$, $w \neq 000, 111$. The number I_w of infinite w -clusters is μ_∞ -a.s. no greater than 1.*
- (b) ([18, Thm 4.4, Prop. 4.7]) *Fix $b, c > 0$. For sufficiently small $a > 0$, there exists μ_∞ -a.s. no infinite type- a cluster. For sufficiently large a , the μ_∞ -probability that the origin belongs to an infinite type- a cluster is strictly positive.*

Part (a) is proved in [19] using an adaptation of the method of Burton and Keane (see [4, 22]) to the 1-2 model, subject to the complication that μ_∞ does not have the so-called ‘finite energy property’. It is unknown whether infinite w -clusters and w' -clusters can coexist with $w \neq w'$.

Part (b) indicates the existence of a phase transition. It is not known if there exists a single critical point $a_c = a_c(b, c)$ for the given property. Furthermore, there is currently no indication of the exact value of such a point.

4.3. Non-analyticity of the two-edge correlation function. For distinct edges $e, f \in \mathbb{E}$, we write

$$\langle \sigma_e \sigma_f \rangle = \lim_{n \rightarrow \infty} \langle \sigma_e \sigma_f \rangle_n,$$

which exists by Theorem 2.1, (see [10, Thm 6.2]). We consider here the asymptotic behaviour of $\langle \sigma_e \sigma_f \rangle$ as $|e - f| \rightarrow \infty$. The behaviour of this limit is unknown in general, but a great deal is known if e and f are related in the ‘diagonal’ manner of the forthcoming assumption (4.1), as illustrated in Figure 4.1.

THEOREM 4.4 ([10, Thm 3.1]). *Let $a, b, c > 0$, and let $e, f \in \mathbb{E}$ be NW/SE edges such that:*

- (4.1) *there exists a path $\ell = \ell(e, f)$ of \mathbb{H}_n from e to f
using only horizontal and NW/SE half-edges.*

- (a) *Let $a \geq b > 0$. For $c > 0$ satisfying*

$$\text{either } \sqrt{a} > \sqrt{b} + \sqrt{c} \quad \text{or} \quad \sqrt{c} > \sqrt{a} + \sqrt{b}$$

except possibly on some set of isolated points, the limit $\lim_{|e-f| \rightarrow \infty} \langle \sigma_e \sigma_f \rangle^2$ exists and is non-zero.

- (b) *If $a \geq b > 0$ and*

$$\sqrt{a} - \sqrt{b} < \sqrt{c} < \sqrt{a} + \sqrt{b},$$

then $\langle \sigma_e \sigma_f \rangle \rightarrow 0$ as $|e - f| \rightarrow \infty$.

A strengthened version of this theorem may be found in [10], allowing a partial relaxation of assumption (4.1), and including a proof of an exponential rate of convergence.

By Theorem 4.4, when $a \geq b \geq c > 0$, the phase transition occurs when $\sqrt{a} = \sqrt{b} + \sqrt{c}$. The proof is along the following lines. The square $\langle \sigma_e \sigma_f \rangle^2$ of the two-edge correlation may be expressed as the determinant of an explicit block Toeplitz matrix (this is where (4.1) is used); see the forthcoming Lemma 4.5. Its limit as $|e - f| \rightarrow \infty$ is given by Widom’s theorem (see [27, 28] and [10, Thm 8.7]) as the determinant of the limiting (infinite) block Toeplitz matrix. This determinant

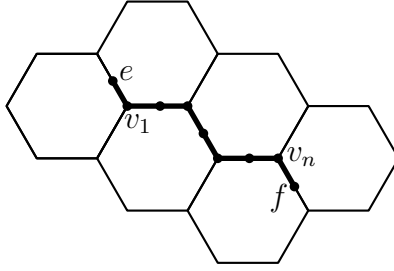


FIGURE 4.1. A path ℓ comprising horizontal and NW/SE mid-edges, connecting the midpoints of two NW/SE edges e and f .

is complex analytic with respect to the parameters a, b, c except when the spectral curve has a unique real zero on the unit torus. As remarked at the end of Section 3.1, the last occurs if and only if $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0$. When $a \geq b \geq c$, this equation becomes $\sqrt{a} - \sqrt{b} - \sqrt{c} = 0$.

The ‘isolated points’ of part (b) arise through the use in the proof of the fact that, for an analytic function Λ , either $\Lambda \equiv 0$ or the zeros of Λ are isolated.

It is easily seen from Figures 2.1 and 4.1 that

$$\langle \sigma_e \sigma_f \rangle = 1 \quad \text{if either } a, b > 0, c = 0, \quad \text{or } a = b = 0, c > 0,$$

and part (b) of Theorem 4.4 follows by the analyticity. For part (a), one uses the representation of the 1-2 model as the Ising model of Section 3.3.

The key step in the proof of Theorem 4.4 is the following exact formula. Let Y_1 be the 2×2 matrix

$$Y_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let Y_{2k} be the $4k \times 4k$ block diagonal matrix with diagonal 2×2 blocks equal to Y_1 . That is,

$$Y_{2k} = \begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_1 \end{pmatrix}.$$

LEMMA 4.5 ([10, Lemma 8.4]). *Suppose the path $\ell = \ell(e, f)$ of (4.1) passes $2k$ bisector edges of \mathbb{H}_Δ . We have that*

$$\langle \sigma_e \sigma_f \rangle = \text{Pf} [Y_{2k} + 2cK_\ell^{-1}],$$

where K_ℓ^{-1} is as in Theorem 4.1.

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