Bond Percolation on Isoradial Graphs

Geoffrey Grimmett

Cambridge University

20 November 2011
Percolation, a model for a random medium

Each edge of $\mathbb{Z}^2$ is declared open with probability $p$.

Open edges transmit water/disease etc.

[1957: Broadbent and Hammersley]
Critical phenomenon

Open connections: $x \leftrightarrow y$

Percolation probability: $\theta(p) = P_p(0 \leftrightarrow \infty)$

Critical point: $p_c = \sup\{p : \theta(p) = 0\}$
**Exact calculation**

**Theorem (Harris 1960 + Kesten 1980)**

The critical point of bond percolation on $\mathbb{Z}^2$ is $p_c = \frac{1}{2}$.

**Conjecture**

For all lattices, $\theta(p_c) = 0$.

**Known:** some two-dimensional lattices, and $\mathbb{Z}^d$ with $d \geq 19$. 
Theorem (Harris 1960 + Kesten 1980)

The critical point of bond percolation on $\mathbb{Z}^2$ is $p_c = \frac{1}{2}$.

Conjecture

For all lattices, $\theta(p_c) = 0$.

Known: some two-dimensional lattices, and $\mathbb{Z}^d$ with $d \geq 19$. 
John Hammersley and Harry Kesten, Oxford, 1993
Kesten, Peierls, Dobrushin in Oxford, 1993
Square, triangular, hexagonal lattices

Homogeneous, inhomogeneous ...
**Box-crossing property**

**Box-crossing property**: For given aspect-ratio $\alpha > 0$, there exists $\delta = \delta(\alpha) > 0$ such that

$$P\left(\begin{array}{c}
N \\
\alpha N
\end{array}\right) \geq \delta,$$

uniformly in $N$ and the position and rotation of the box.

[Russo–Seymour–Welsh, Kesten]
Rhombic tiling

[Penrose, de Bruijn]
Rhombic tiling + isoradial graph

[Duffin +]
Examples of track-systems

Square lattice

Penrose tiling

Characterization (de Bruijn, Kenyon–Schlenker):

- No track intersects itself
- Two tracks have \( \leq 1 \) intersection
Examples of track-systems

Square lattice  Penrose tiling

Characterization (de Bruijn, Kenyon–Schlenker):

• No track intersects itself
• Two tracks have \( \leq 1 \) intersection
Penrose percolation
Requisite properties of track-systems

Bounded-angles property (BAC): Angles of rhombi are bounded away from 0 and $\pi$

Square-grid property (SQP): The track-system contains a square grid (and boundedly many intersections of tracks between grid-intersections)

$\mathcal{G} = \{\text{all isoradial } G \text{ with the BAC and the SGP}\}$
A track-system without the SGP
Canonical percolation measure

\[
\frac{p_e}{1 - p_e} = \frac{\sin\left(\frac{1}{3}[\pi - \theta_e]\right)}{\sin\left(\frac{1}{3}\theta_e\right)}.
\]

isoradial $G$ $\longleftrightarrow$ percolation measure $\mathbb{P}_G$

dual graph $G^*$ $\longleftrightarrow$ dual measure $\mathbb{P}_{G^*}$

[Kenyon]
Criticality and Universality

Subject to formal ratification

Theorem (G+Manolescu 2011)

Let $G \in \mathcal{G}$.

1. $P_G$ (and $P_{G^*}$) has the box-crossing property.
2. $P_G$ is critical.
3. $\mathcal{G}$ is a universality class, in that $\rho, \eta (+ \ldots)$ are constant over $\mathcal{G}$ (assuming they exist for some $G \in \mathcal{G}$).

Near-critical exponents: Following Kesten (1987), one needs further regularity on the graph.
Subject to formal ratification

**Theorem (G+Manolescu 2011)**

Let $G \in \mathcal{G}$.

1. $\mathbb{P}_G$ (and $\mathbb{P}_{G^*}$) has the box-crossing property.
2. $\mathbb{P}_G$ is critical.
3. $\mathcal{G}$ is a universality class, in that $\rho$, $\eta$ (+ . . .) are constant over $\mathcal{G}$ (assuming they exist for some $G \in \mathcal{G}$).

Near-critical exponents: Following Kesten (1987), one needs further regularity on the graph.
Subject to formal ratification

Theorem (G+Manolescu 2011)

Let $G \in \mathcal{G}$.

1. $\mathbb{P}_G$ (and $\mathbb{P}_{G^*}$) has the box-crossing property.
2. $\mathbb{P}_G$ is critical.
3. $\mathcal{G}$ is a universality class, in that $\rho$, $\eta$ (+ ...) are constant over $\mathcal{G}$ (assuming they exist for some $G \in \mathcal{G}$).

Near-critical exponents: Following Kesten (1987), one needs further regularity on the graph.
Critical exponents

$C$ is the open cluster at vertex $v$.

$$
P_G(|C| = n) \approx n^{-1-1/\delta}
$$

$$
P_G(\text{rad}(C) = n) \approx n^{-1-1/\rho}
$$

$$
P_G(0 \leftrightarrow x) \approx |x|^{2-d-\eta}
$$

Scaling relations with $d = 2$: $\eta\rho = 2$, $2\rho = \delta + 1$. 
Critical exponents

$C$ is the open cluster at vertex $v$.

\[ P_G(|C| = n) \approx n^{-1 - 1/\delta} \]
\[ P_G(\text{rad}(C) = n) \approx n^{-1 - 1/\rho} \]
\[ P_G(0 \leftrightarrow x) \approx |x|^{2 - d - \eta} \]

Scaling relations with $d = 2$: $\eta \rho = 2$, $2 \rho = \delta + 1$. 
Arm exponents

Alternating arm exponents:
\[ \mathbb{P}_G \left( j \text{ primal arms interspersed with } j \text{ dual arms} \right) \approx n^{-\rho_{2j}} \]

One- and four-arm exponents: \( \rho_1 \) and \( \rho_4 \) are especially useful.
Arm exponents

\[ P_G \left( n \right) \approx n^{-\rho_6} \]

Alternating arm exponents:
\[ P_G \left( j \right \text{ primal arms interspersed with } j \text{ dual arms} \right) \approx n^{-\rho_{2j}} \]

One- and four-arm exponents: \( \rho_1 \) and \( \rho_4 \) are especially useful.
Exact values in two dimensions

Theorem (Smirnov, Werner, Kesten, +)

For site percolation on the triangular lattice,

\[ \beta = \frac{5}{36}, \quad \gamma = \frac{43}{18}, \quad \delta = \frac{91}{5}, \quad \rho = \frac{48}{5}, \]

\[ \rho_{2j} = \frac{1}{12} (4j^2 - 1). \]

Questions:

- \( \alpha = -\frac{2}{3} \)?
- other lattices in two dimensions, e.g. \( \mathbb{Z}^2 \) bond or site?
- Universality?
Exact values in two dimensions

Theorem (Smirnov, Werner, Kesten, +)

For site percolation on the triangular lattice,

\[ \beta = \frac{5}{36}, \quad \gamma = \frac{43}{18}, \quad \delta = \frac{91}{5}, \quad \rho = \frac{48}{5}, \]

\[ \rho_{2j} = \frac{1}{12}(4j^2 - 1). \]

Questions:

• \( \alpha = -\frac{2}{3} \) ?

• other lattices in two dimensions, e.g. \( \mathbb{Z}^2 \) bond or site?

• Universality?
Connectivity within \( \{A, B, C\} \) is preserved if \( \kappa(p_0, p_1, p_2) = 0 \), where

\[
\kappa(p_0, p_1, p_2) = p_0 + p_1 + p_2 - p_0p_1p_2 - 1.
\]
**Coupling**

\[
P := (1 - p_0)(1 - p_1)(1 - p_2)
\]
Star-triangle via tracks
Star–triangle via tracks

Rhombus switch = star-triangle transformation
= operation on tracks
Isoradial square lattice with associated rhombi and tracks.
Swapping tracks

irregular

semi-regular
Transporting paths

Probabilistic estimates are required to control path-geometry
The general case, square-grid property

[Kenyon +?]
Conclusions

Bond percolation on many isoradial graphs:

- The isoradial embedding produces critical percolation
- Many critical exponents are invariant within this class, if they exist.
- Site percolation on $T$: Conformality, SLE, critical exponents exist and take their predicted values,
Conclusions

Bond percolation on many isoradial graphs:

- The isoradial embedding produces critical percolation
- Many critical exponents are invariant within this class, if they exist.
- **Site percolation on $\mathbb{T}$:** Conformality, SLE, critical exponents exist and take their predicted values,
Further directions

- Star–triangle transformations preserve interfaces
- Find a connection to site percolation on $\mathbb{T}$
- Remove ‘square-grid assumption’
- Prove universality of near-critical exponents
- Random-cluster model?

[Chelkak–Smirnov]
Further directions

- Star–triangle transformations preserve interfaces
- Find a connection to site percolation on $\mathbb{T}$
- Remove ‘square-grid assumption’
- Prove universality of near-critical exponents
- Random-cluster model?

[Chelkak–Smirnov]
Further directions

• Star–triangle transformations preserve interfaces
• Find a connection to site percolation on $\mathbb{T}$
• Remove ‘square-grid assumption’
• Prove universality of near-critical exponents
• Random-cluster model?

[Chelkak–Smirnov]
Further directions

- Star–triangle transformations preserve interfaces
- Find a connection to site percolation on $\mathbb{T}$
- Remove ‘square-grid assumption’
- Prove universality of near-critical exponents
  - Random-cluster model?

[Chelkak–Smirnov]
Further directions

- Star–triangle transformations preserve interfaces
- Find a connection to site percolation on $\mathbb{T}$
- Remove ‘square-grid assumption’
- Prove universality of near-critical exponents
- Random-cluster model?

[Chelkak–Smirnov]