CHAPTER 9
Spatial Processes

In this chapter we shall be interested in processes $\mathbf{n}=(n_1, n_2, \ldots, n_J)$ capable of modelling systems containing a finite number of sites or components. The idea is that n_j describes the attribute (or state) of site j and that changes in this attribute are affected by the attributes of sites adjacent to site j. For example sites might be fruit trees in an orchard, and n_j might take the value of unity or zero depending on the presence or absence of disease. In previous chapters the equilibrium distributions obtained have often implied the independence of $n_1, n_2, \ldots, n_J$. Some of the models considered in this chapter lead to a more complicated equilibrium distribution in which there is a limited dependence between $n_1, n_2, \ldots, n_J$. Before discussing these models we shall, in Section 9.1, make precise this concept of limited dependence. Later, in Section 9.4, we shall use the setting provided by these models to discuss the relationship between partial balance and insensitivity.

9.1 MARKOV FIELDS

Consider a system consisting of J sites, each of which has associated with it an attribute. Let n_j be the attribute of site j, where n_j is chosen from a set $\mathcal{N}_j$, assumed for simplicity to be finite. Thus the state of the system, $\mathbf{n}=(n_1, n_2, \ldots, n_J)$, takes values in the state space $\mathcal{S}=\mathcal{N}_1\times\mathcal{N}_2\times\cdots\times\mathcal{N}_J$. A function

$$\pi: \mathcal{S}\rightarrow(0, 1)$$

is called a random field if

$$\sum_{\mathbf{n}\in\mathcal{S}}\pi(\mathbf{n})=1$$

Thus a random field is just a probability distribution over the state space of the system which assigns a positive probability to every state.

To specify the spatial relationship between the sites we shall use a graph theoretic framework. Suppose the J sites of the system are the vertices of a graph G, and call sites j and k neighbours if they are joined by an edge of the graph. Write ∂j for the set of neighbours of j. The same symbol will be used for a graph and its vertex set, and for a site and the set consisting of just that site; thus $G-j$ will refer to the set of sites other than j. For any $H\subset G$ let $|H|$ be the number of sites in H and let $\mathbf{n}_H$ be the $|H|$-dimensional vector giving the value associated with each site in H. Let T_j^m be the operator which changes the attribute of site j to m. Thus

$$T_j^m\mathbf{n}=(n_1, n_2, \ldots, n_{j-1}, m, n_{j+1}, \ldots, n_J)$$

Given a random field π we can calculate the conditional distribution of n_j given the attributes of some or all of the other sites of the system. For example the conditional probability that site j has attribute n_j given that the other sites have attributes $\mathbf{n}_{G-j}$ is

$$P(n_j \mid \mathbf{n}_{G-j})=\frac{\pi(\mathbf{n})}{\sum_{m\in\mathcal{N}_j}\pi(T_j^m\mathbf{n})} \tag{9.1}$$

Conversely for a finite graph G the random field π can be calculated from the conditional probabilities.

Lemma 9.1. *The conditional probabilities $P(n_j \mid \mathbf{n}_{G-j})$, $j\in G$, $\mathbf{n}\in\mathcal{S}$, determine uniquely the random field $\pi(\mathbf{n})$, $\mathbf{n}\in\mathcal{S}$.*

Proof. Let 0 denote one of the attributes from each of the sets $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_J$, and let $\mathbf{0}=(0, 0, \ldots, 0)$. Observe that the conditional probabilities $P(n_j \mid \mathbf{n}_{G-j})$ determine the ratios $\pi(\mathbf{n})/\pi(T_j^m\mathbf{n})$ since

$$\frac{P(n_j \mid \mathbf{n}_{G-j})}{P(m \mid \mathbf{n}_{G-j})}=\frac{\pi(\mathbf{n})}{\pi(T_j^m\mathbf{n})}$$

Now

$$\frac{\pi(\mathbf{n})}{\pi(\mathbf{0})}=\frac{\pi(\mathbf{n})}{\pi(T_1^0\mathbf{n})}\frac{\pi(T_1^0\mathbf{n})}{\pi(T_1^0T_2^0\mathbf{n})}\cdots\frac{\pi(T_1^0T_2^0\cdots T_{J-1}^0\mathbf{n})}{\pi(T_1^0T_2^0\cdots T_J^0\mathbf{n})}$$

and hence this ratio is also determined by the conditional probabilities. The probability $\pi(\mathbf{0})$ can be deduced from the normalization condition $\sum\pi(\mathbf{n})=1$, and hence the result is proved.

The conditional probabilities $P(n_j \mid \mathbf{n}_{G-j})$ cannot be chosen arbitrarily. They must satisfy certain consistency conditions: the ratio $\pi(\mathbf{n})/\pi(\mathbf{0})$ must not depend on the particular sequence of states $\mathbf{n}$, $T_1^0\mathbf{n}$, $T_1^0T_2^0\mathbf{n}, \ldots, \mathbf{0}$ used to calculate it.

In general $P(n_j \mid \mathbf{n}_{G-j})$ will depend upon the entire vector $\mathbf{n}_{G-j}$, but occasionally it may depend on only some of the components of this vector. Call a random field a *Markov field* if

$$P(n_j \mid \mathbf{n}_{G-j})=P(n_j \mid \mathbf{n}_{\partial j}) \tag{9.2}$$

Thus with a Markov field the conditional probability distribution for the attribute of site j given the attributes of all other sites in the system depends only upon the attributes of sites which are neighbours of site j. Of course if every pair of sites in G is connected by an edge then equation (9.2) is always satisfied. At the other extreme if G contains no edges then condition (9.2) implies that $n_1, n_2, \ldots, n_J$ are independent.

An example of a Markov field is provided if $n_1, n_2, \ldots, n_J$ form successive observations from a realization of a Markov chain. Then

$$\pi(n_1, n_2, \ldots, n_J) = P(n_1)p(n_1, n_2)p(n_2, n_3) \cdots p(n_{J-1}, n_J)$$

and so

$$\begin{aligned} P(n_j \mid \mathbf{n}_{G-j}) &= \frac{p(n_{j-1}, n_j)p(n_j, n_{j+1})}{\sum_m p(n_{j-1}, m)p(m, n_{j+1})} \\ &= P(n_j \mid n_{\partial j}) \qquad 2 \le j \le J-1 \end{aligned}$$

provided we identify the neighbours of j as $j-1$ and $j+1$. This example explains why we use the term Markov field. For a Markov chain $n_1, n_2, \ldots, n_J$ are usually regarded as observations taken at different points in time; here we prefer to regard $n_1, n_2, \ldots, n_J$ as observations taken at different points in space. From a temporal viewpoint the relation

$$P(n_j \mid n_1, n_2, \ldots, n_{j-1}) = P(n_j \mid n_{j-1}) \tag{9.3}$$

is the most natural definition of a Markov chain; we shall see that this is equivalent to the relation

$$P(n_j \mid n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_J) = P(n_j \mid n_{j-1}, n_{j+1}) \tag{9.4}$$

The main result of this section is the next theorem which establishes the form a random field must take if it is to be Markov. To state the theorem we need a little more notation. Call a subset $C \subseteq G$ a simplex if an edge joins any two distinct sites in C, or if C consists of just one site. Let $\mathscr{C}$ denote the set of simplices of the graph G.

Theorem 9.2. *A random field π is a Markov field if and only if it can be written in the form*

$$\pi(\mathbf{n}) = B \prod_{C \in \mathscr{C}} \phi_C(\mathbf{n}_C) \qquad \mathbf{n} \in \mathscr{S} \tag{9.5}$$

Proof. Suppose that π is a Markov field. Write $\mathbf{n}_H^0$ for the J-dimensional vector whose jth component is n_j if $j \in H$ and is zero otherwise. Define the functions $\phi_C(\mathbf{n}_C)$, $C \in \mathscr{C}$, by the recursion

$$\begin{aligned} \phi_j(n_j) &= \pi(\mathbf{n}_j^0) \\ \phi_C(\mathbf{n}_C) &= \frac{\pi(\mathbf{n}_C^0)}{\prod_{H \subset C} \phi_H(\mathbf{n}_H)} \end{aligned} \tag{9.6}$$

Thus $\pi(\mathbf{n})$ takes the form (9.5) with $B=1$ whenever $\mathbf{n}=\mathbf{n}_C^0$ for some $C\in\mathscr{C}$; to prove it always takes this form we shall work by induction on the number of non-zero components in $\mathbf{n}$. If $\mathbf{n}$ is not equal to $\mathbf{n}_C^0$ for some $C\in\mathscr{C}$ then there must be sites j and k which are not neighbours such that n_j and n_k are non-zero. Since π is a Markov field the ratio $\pi(\mathbf{n})/\pi(T_k^0\mathbf{n})$ does not depend upon n_j. Hence

$$\frac{\pi(\mathbf{n})}{\pi(T_k^0\mathbf{n})}=\frac{\pi(T_j^0\mathbf{n})}{\pi(T_j^0T_k^0\mathbf{n})}$$

Write this in the alternative form

$$\pi(\mathbf{n})=\frac{\pi(T_j^0\mathbf{n})\,\pi(T_k^0\mathbf{n})}{\pi(T_j^0T_k^0\mathbf{n})} \tag{9.7}$$

Now $T_j^0\mathbf{n}$, $T_k^0\mathbf{n}$, and $T_j^0T_k^0\mathbf{n}$ all have more zero components than $\mathbf{n}$. Hence our inductive hypothesis allows us to assume $\pi(T_j^0\mathbf{n})$, $\pi(T_k^0\mathbf{n})$, and $\pi(T_j^0T_k^0\mathbf{n})$ are of the form (9.5) with $B=1$. Substitution into equation (9.7) establishes that $\pi(\mathbf{n})$ is also of this form, and so the induction is complete.

The converse is simple to prove. If $\pi(\mathbf{n})$ is of the form (9.5) then

$$\begin{aligned}P(n_j\mid \mathbf{n}_{G-j})&=\frac{\prod_{C\in\mathscr{C}:j\in C}\phi_C(\mathbf{n}_C)}{\sum_{m\in\mathcal{N}_j}\prod_{C\in\mathscr{C}:j\in C}\phi_C(T_j^m\mathbf{n}_C)}\\&=P(n_j\mid \mathbf{n}_{\partial j})\end{aligned}$$

To illustrate the theorem consider the case where the graph G is a finite region of a rectangular lattice and the field is binary, i.e. the attributes of sites are either zero or unity. A site has at most four neighbours, and the only simplices are single sites and pairs of adjacent sites. Suppose that site 1 is internal to the lattice and that its four neighbours are sites 2, 3, 4, and 5. If π is a Markov field the conditional distribution $P(n_1\mid n_{G-1})$ is determined by the ratio

$$\frac{P(n_1=1\mid n_2,n_3,n_4,n_5)}{P(n_1=0\mid n_2,n_3,n_4,n_5)}=\frac{\phi_1(1)\phi_{\{1,2\}}(1,n_2)\phi_{\{1,3\}}(1,n_3)\phi_{\{1,4\}}(1,n_4)\phi_{\{1,5\}}(1,n_5)}{\phi_1(0)\phi_{\{1,2\}}(0,n_2)\phi_{\{1,3\}}(0,n_3)\phi_{\{1,4\}}(0,n_4)\phi_{\{1,5\}}(0,n_5)}$$

Let α_1 be the value of this ratio when $n_2=n_3=n_4=n_5=0$, and let

$$\beta_{\{1,k\}}=\frac{\phi_{\{1,k\}}(0,0)\phi_{\{1,k\}}(1,1)}{\phi_{\{1,k\}}(1,0)\phi_{\{1,k\}}(0,1)}$$

Then

$$\frac{P(n_1=1\mid n_2,n_3,n_4,n_5)}{P(n_1=0\mid n_2,n_3,n_4,n_5)}=\alpha_1\prod_{k=2}^{5}\beta_{\{1,k\}}^{n_k}$$

Thus a binary Markov field on a rectangular lattice is determined by a relatively small set of parameters: one for each site and one for each pair of adjacent sites. Often symmetry considerations reduce this even further to just two parameters α and β. For an internal site j the conditional probabilities can then be written

$$P(n_j = 1 \mid \mathbf{n}_{G-j}) = \frac{\alpha\beta^r}{1+\alpha\beta^r} \tag{9.8}$$

where r is the number of sites neighbouring j whose attribute is unity. The simplest way to deal with edge effects is to suppose that each lattice point neighbouring the region G is a site which has a known attribute, either zero or unity; expression (9.8) can then be taken to define the conditional probabilities even when j is a site on the boundary of the region, with r including any neighbouring site outside the region whose attribute is unity. The field π can be written as

$$\pi(\mathbf{n}) = B\alpha^M\beta^R \tag{9.9}$$

where $M = \sum n_j$ and R is the number of pairs of neighbouring sites in which the attributes of both sites of the pair are unity.

Exercises 9.1

1. From the recursion (9.6) deduce Grimmett's formula

$$\phi_C(\mathbf{n}_C) = \exp\left(\sum_{H \subseteq C} (-1)^{|C-H|} \log \pi(\mathbf{n}_H^0)\right)$$

2. If $\pi(\mathbf{n})$ is a Markov field the functions ϕ_C appearing in the form (9.5) can be chosen in various ways. For the Markov chain example the obvious choice is

$$\begin{aligned} \phi_1(n_1) &= P(n_1) & \\ \phi_j(n_j) &= 1 & 2 \le j \le J \\ \phi_{\{j,j+1\}}(n_j, n_{j+1}) &= p(n_j, n_{j+1}) & 1 \le j \le J-1 \end{aligned}$$

Check that for $2 \le j \le J-1$ Grimmett's formula gives

$$\phi_j(n_j) = P(0)p(0,0)^{J-3}p(0,n_j)p(n_j,0)$$

$$\phi_{\{j,j+1\}}(n_j, n_{j+1}) = \frac{p(n_j, n_{j+1})}{P(0)p(0,0)^{J-2}p(n_j,0)p(0,n_{j+1})}$$

3. Check that for a sequence of random variables $n_1, n_2, \ldots, n_J$ relations (9.3) and (9.4) are equivalent; assume that n_0 and n_{J+1} take known values.
4. Consider a binary Markov field on a graph G which has no simplices

containing more than two sites. Show that if

$$P(n_j = 1 \mid \mathbf{n}_{G-j}) = p_r$$

for all j and $\mathbf{n}_{G-j}$, where r is the number of sites neighbouring j whose attribute is unity, then

$$p_r = \frac{\alpha\beta^r}{1+\alpha\beta^r}$$

for some α and β.

5. Show that for a Markov field the probability distribution $P(\mathbf{n}_H \mid \mathbf{n}_{G-H})$ depends on $\mathbf{n}_{G-H}$ only through the attributes of sites neighbouring H. Associate a graph with the subset H by deleting from the graph G all the sites in $G-H$ and all the edges emanating from these sites. Show that the probability distribution $P(\mathbf{n}_H \mid \mathbf{n}_{G-H})$ is a Markov field over the graph H. Deduce that if H is a tree in which no site has more than two neighbours then, conditional on $\mathbf{n}_{G-H}$, $\mathbf{n}_H$ is a Markov chain. If G is the union of three disjoint sets H_1, H_2, H_3 and if no edge of G joins a site in H_1 to a site in H_3 show that, conditional on $\mathbf{n}_{H_2}$, the random vectors $\mathbf{n}_{H_1}$ and $\mathbf{n}_{H_3}$ are independent.

9.2 REVERSIBLE SPATIAL PROCESSES

Under what conditions will the equilibrium distribution $\pi(\mathbf{n})$ of a stochastic process $\mathbf{n}(t)$ be a Markov field? If changes of attribute at site j are influenced only by the attributes of sites neighbouring j then we might hope that $\pi(\mathbf{n})$ would be a Markov field. However, the transition rates of the invasion processes considered in Section 5.3 have this local character and yet their equilibrium distributions are not Markov fields (Exercise 9.2.1). That a field $\pi(\mathbf{n})$ is Markov is an attractive assumption to make but it is not justified solely by the local character of the transition rates of $\mathbf{n}(t)$. Further restrictions are necessary.

Call $\mathbf{n}(t)$ a *spatial process* if:

(i) Only one component of $\mathbf{n}$ can change at a time.

(ii) The transition rate $q(\mathbf{n}, T_j^m\mathbf{n})$ does not depend on $\mathbf{n}_{G-j-\partial j}$.

(iii) For any states $\mathbf{n}$, $T_j^m\mathbf{n}$ it is possible to reach $T_j^m\mathbf{n}$ from $\mathbf{n}$ by a sequence of transitions which do not alter $\mathbf{n}_{G-j}$.

Condition (iii) can be viewed as a strengthened version of the usual irreducibility assumption. The invasion processes of Section 5.3 are spatial processes provided ν_j and μ_j are positive for all j.

Theorem 9.3. *The equilibrium distribution of a reversible spatial process is a Markov field.*

Proof. The equilibrium distribution $\pi(\mathbf{n})$ satisfies the detailed balance condition

$$\pi(\mathbf{n})q(\mathbf{n}, T_j^m\mathbf{n}) = \pi(T_j^m\mathbf{n})q(T_j^m\mathbf{n}, \mathbf{n})$$

Hence if $q(\mathbf{n}, T_j^m\mathbf{n}) > 0$ the ratio $\pi(\mathbf{n})/\pi(T_j^m\mathbf{n})$ does not depend upon $\mathbf{n}_{G-j-\partial j}$. Condition (iii) ensures that this is true even if it takes more than one transition to reach the state $T_j^m\mathbf{n}$ from $\mathbf{n}$. But these ratios, for $m \in \mathcal{N}_j$, determine the conditional distribution $P(n_j \mid \mathbf{n}_{G-j})$, which is thus equal to $P(n_j \mid \mathbf{n}_{\partial j})$. Hence $\pi(\mathbf{n})$ is a Markov field.

To illustrate the theorem we shall discuss some examples of spatial processes. Suppose that $\mathcal{N}_j = \mathcal{N} = \{1, 2, \ldots, N\}$ and that

$$q(\mathbf{n}, T_j^m\mathbf{n}) = \lambda(n_j, m)\phi(n_j)^r\psi(m)^{r'} \tag{9.10}$$

where r and r' are the numbers of sites neighbouring j which have attributes n_j and m respectively. We can regard $\lambda(n_j, m)$ as the innate tendency of a site's attribute to change from n_j to m, and $\phi(n_j)$ and $\psi(m)$ as measures of the extent to which this tendency is increased or decreased by the existence of neighbouring sites with attributes n_j or m. For example sites may be individuals and attributes may be views on a subject, to give a setting which may help visualize the process. Kolmogorov's criteria readily show that the process is reversible if and only if the rates $\lambda(n, m)$ define a reversible process on the state space $\mathcal{N}$, which happens if and only if there exists a non-zero solution $\alpha(n)$, $n = 1, 2, \ldots, N$, to the equations

$$\alpha(n)\lambda(n, m) = \alpha(m)\lambda(m, n) \tag{9.11}$$

When this is so the equilibrium distribution for the process $\mathbf{n}(t)$ is

$$\pi(\mathbf{n}) = B\prod_{n=1}^{N} \alpha(n)^{M(n)}\left[\frac{\psi(n)}{\phi(n)}\right]^{R(n)} \tag{9.12}$$

where $M(n)$ is the number of sites with attribute n and $R(n)$ is the number of n-bonds, i.e. edges of the graph G which have sites with attribute n at both ends. When $N = 2$ equations (9.11) must have a solution and the equilibrium distribution can be rewritten in the form (9.9).

A drawback of the above model is that the dependence of the transition rates on r, r' is restricted to the multiplicative form given in expression (9.10). The adjective multiplicative is used since if r is increased by one the rate is multiplied by a factor. This form of dependence is typical of processes which have Markov fields as their equilibrium distributions; for reversible processes it is generally a consequence of the detailed balance condition taken together with the multiplicative form enforced by Theorem 9.2.

It is interesting to note that if the functions ϕ, ψ take values close to unity then the rates (9.10) take an approximately additive form. For example suppose $N = 2$, $\lambda(1, 0) = \lambda$, $\lambda(0, 1) = \mu$, $\phi(0) = \phi(1) = \psi(0) = 1$, $\psi(1) = 1 + \delta$.

Consider this process as a model for the ebb and flow of a recurrent infection over an array of plants, with zero indicating a healthy plant and unity indicating the presence of disease. The rate at which an infected plant recovers is λ, and the rate at which a healthy plant becomes infected is

$$\mu(1+\delta)^{r'}$$

where r' is the number of neighbouring plants which are infected. If δ is small this infection rate is approximately equal to

$$\mu+\mu\delta r' \tag{9.13}$$

This is the form we would expect if plants are infected by germs which come from the general environment with intensity μ and from an adjacent infected plant with intensity $\mu\delta$. The approximation will thus be reasonable if germs from the general environment are a significant source of infection. The equilibrium distribution is

$$\pi(\mathbf{n})=B\left(\frac{\mu}{\lambda}\right)^{M(1)}(1+\delta)^{R(1)} \tag{9.14}$$

Our next example of a spatial process has a rather different setting. Suppose the sites are power sources that are connected to power users in such a way that each user has two possible sources of power. Represent the users served by sources j and k as an edge joining sources j and k. Let $d(j, k)$ be the amount of power required by these users; define $d(j, k)$ to be zero if there is no user served by sources j and k. Let n_j be unity or zero according to whether source j is functioning or broken down. If sources j and k are both broken down then demand $d(j, k)$ is unsatisfied. If one of the sources j or k is functioning it supplies the entire demand $d(j, k)$, while if both sources are functioning they each supply an amount $\frac{1}{2}d(j, k)$. Thus if source j is functioning it carries a load $\frac{1}{2}\sum_k(1+n_k)d(j, k)$. If $n_j=1$ let

$$q(\mathbf{n}, T_j^0\mathbf{n})=\lambda_j\exp\left(\tfrac{1}{2}\gamma\sum_k(1+n_k)d(j, k)\right)$$

This breakdown rate corresponds to the fairly severe assumption that each additional unit of load a source has to carry increases its failure rate by a factor e^{γ}. If $n_j=0$ let

$$q(\mathbf{n}, T_j^1\mathbf{n})=\mu_j$$

so that source j remains broken down for a period exponentially distributed with mean μ_j^{-1}. It is readily checked that the detailed balance conditions hold with

$$\pi(\mathbf{n})=B\left\{\prod_{j=1}^{J}\left(\frac{\mu_j}{\lambda_j}\right)^{n_j}\right\}\exp\left(-\tfrac{1}{2}\gamma\sum_{1\le j<k\le J}(1+n_j)(1+n_k)d(j, k)\right)$$

With respect to the graph G formed by linking sources j and k if $d(j, k)>0$ the process $\mathbf{n}$ is a reversible spatial process and the equilibrium distribution is a Markov field. Exercise 9.2.4 discusses a natural extension of this model in which the resulting Markov field involves factors arising from simplices containing more than two sites.

Exercises 9.2

1. For the invasion process of Section 5.3 let $n_j = 0$ or 1 depending on whether site j is white or black. Show that the equilibrium distribution $\pi(\mathbf{n})$ of an invasion process is not in general a Markov field, even when ν_j and μ_j are positive for all j.
2. Our definition of a random field required that it assign positive probability to every state in $\mathscr{S}$. If we remove this restriction we might hope that a field satisfying (9.2) whenever the conditioning events have positive probabilities could be expressed in the form (9.5) or as a limiting case of this form with some of the functions ϕ approaching zero or infinity. This is not so; there exist counterexamples. Similarly, the strong irreducibility condition (iii) is more than a restriction introduced to simplify the proof of Theorem 9.3. Let the graph G consist of three sites with edges joining sites 1 and 2 and sites 2 and 3, and let $\mathcal{N} = \{0, 1\}$. Consider the process (n_1, n_2, n_3) with transition rates as given in Fig. 9.1. Observe that it is a reversible process satisfying conditions (i) and (ii) but not (iii). Show that its equilibrium distribution is not a Markov field.

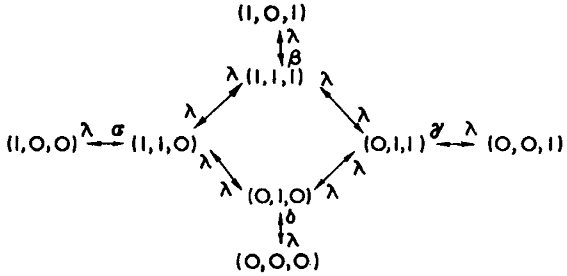

Fig. 9.1 A reversible process whose equilibrium distribution is not a Markov field

3. Consider the plant infection model described in the preceding section. Observe that even with the additive form (9.13) the model is not an invasion model, since healthy plants do not encourage the recovery of adjacent infected plants. Consider now the following elaboration of the model. Suppose that while plant j is infected germs destined to infect plant k are emitted from it at rate $\mu\delta_{jk}$, where $\delta_{jk} = \delta_{kj}$. The value of δ_{jk} might depend on the distance between plants j and k, and could possibly be zero. Show that provided the δ's are small the model approximates a process whose equilibrium distribution is

$$\pi(\mathbf{n}) = B\left(\frac{\mu}{\lambda}\right)^{\Sigma n_j} \prod_{1 \le j < k \le J} (1 + \delta_{jk})^{n_j n_k}$$

4. In the power supply model it was assumed that each user had exactly two sources of power. This assumption can be relaxed. Suppose that for each simplex $C \in \mathscr{C}$ of the graph G a demand $d(C)$ arises from users who can take power from any of the sources in C, and that this demand is shared equally over those sources in C which are functioning. Write down the breakdown rate for source j and deduce that in equilibrium

$$\pi(\mathbf{n}) = B\left\{\prod_{j=1}^{J}\left(\frac{\mu_j}{\lambda_j}\right)^{n_j}\right\}\exp\left[-\gamma \sum_{C\in\mathscr{C}}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{\sum_{k\in C} n_k}\right)d(C)\right]$$

9.3 A GENERAL SPATIAL PROCESS

It is easy to construct reversible spatial processes with a given Markov field as their equilibrium distribution, using the detailed balance condition. In this section we shall describe a fairly general process which gives some insight into the way a Markov field can arise as the equilibrium distribution of a non-reversible process. The closed migration process of Chapter 2 is a special case of the process to be described, and in the next section we shall see how this relationship clarifies the phenomenon of partial balance observed in Chapter 2.

Suppose there are defined positive functions $\Phi(\mathbf{n})$, $\Phi_{G-j}(\mathbf{n}_{G-j})$, $j \in G$. Consider the process $\mathbf{n}(t)$ with transition rates

$$q(\mathbf{n}, T_j^m\mathbf{n}) = \lambda_j(n_j, m)\frac{\Phi(\mathbf{n})}{\Phi_{G-j}(\mathbf{n}_{G-j})} \tag{9.15}$$

We can regard $\lambda_j(n_j, m)$ as the innate tendency of site j to change its attribute from n_j to m, and the other term appearing in the rate (9.15) as a measure of the extent to which this is affected by the attributes of the other sites in the system. We shall assume that for each j the equations

$$\alpha_j(n)\sum_{m\in\mathscr{N}_j}\lambda_j(n, m) = \sum_{m\in\mathscr{N}_j}\alpha_j(m)\lambda_j(m, n) \qquad n \in \mathscr{N}_j \tag{9.16}$$

have a positive solution for $\alpha_j(n)$, $n \in \mathscr{N}_j$, which is unique up to a multiplying factor—this is equivalent to the assumption that the state space $\mathscr{S}$ of the process $\mathbf{n}(t)$ is irreducible.

Theorem 9.4. *The equilibrium distribution for the process $\mathbf{n}(t)$ with transition rates (9.15) is*

$$\pi(\mathbf{n}) = B\frac{\prod_{j=1}^{J}\alpha_j(n_j)}{\Phi(\mathbf{n})} \tag{9.17}$$

where B is a normalizing constant.

Proof. The equilibrium equations are

$$\pi(\mathbf{n}) \sum_j \sum_m q(\mathbf{n}, T_j^m \mathbf{n}) = \sum_j \sum_m \pi(T_j^m \mathbf{n}) q(T_j^m \mathbf{n}, \mathbf{n}) \qquad \mathbf{n} \in \mathcal{S}$$

By substitution we can verify that $\pi(\mathbf{n})$ satisfies the partial balance equations

$$\pi(\mathbf{n}) \sum_m q(\mathbf{n}, T_j^m \mathbf{n}) = \sum_m \pi(T_j^m \mathbf{n}) q(T_j^m \mathbf{n}, \mathbf{n}) \qquad \mathbf{n} \in \mathcal{S} \tag{9.18}$$

for each $j \in G$. The equilibrium equations follow from these.

If the function Φ has the appropriate form then $\pi(\mathbf{n})$ will be a Markov field. Whether the process $\mathbf{n}(t)$ is a spatial process will depend on the graph G and on the functions Φ_{G-j}, $j \in G$, as well as on Φ. For example if

$$\Phi(\mathbf{n}) = \prod_{C \in \mathscr{C}} \phi_C(\mathbf{n}_C)$$

$$\Phi_{G-j}(\mathbf{n}_{G-j}) = \prod_{C \in \mathscr{C} : j \in C} \phi_C(\mathbf{n}_C)$$

then

$$q(\mathbf{n}, T_j^m \mathbf{n}) = \lambda_j(n_j, m) \prod_{C \in \mathscr{C} : j \notin C} \phi_C(\mathbf{n}_C) \tag{9.19}$$

so that $\mathbf{n}(t)$ is a spatial process and the equilibrium distribution is the Markov field

$$\pi(\mathbf{n}) = B \frac{\prod_{j=1}^J \alpha_j(n_j)}{\prod_{C \in \mathscr{C}} \phi_C(\mathbf{n}_C)} \tag{9.20}$$

Specializing further suppose $\mathcal{N}_j = \{1, 2, \ldots, N\}$,

$$\phi_C(\mathbf{n}_C) = \begin{cases} \phi(n) & \text{if } |C| = 2 \text{ and } \mathbf{n}_C = (n, n) \\ 1 & \text{otherwise} \end{cases}$$

and $\lambda_j(n, m) = \lambda(n, m)$, so that $\alpha_j(n) = \alpha(n)$. Then

$$q(\mathbf{n}, T_j^m \mathbf{n}) = \lambda(n_j, m) \phi(n_j)^r \tag{9.21}$$

where r is the number of sites neighbouring site j which have the same attribute as site j. We could perhaps regard neighbouring sites with the same attribute as forming a bond which decreases, or increases, the rate of change of attribute at those sites. The equilibrium distribution for the process is

$$\pi(\mathbf{n}) = B \prod_{n=1}^N \frac{\alpha(n)^{M(n)}}{\phi(n)^{R(n)}} \tag{9.22}$$

where $M(n)$ is the number of sites with attribute n and $R(n)$ is the number of n-bonds, i.e. edges of the graph G which have sites with attribute n at both ends.

The process with transition rates (9.21) and equilibrium distribution (9.22) is very similar to the process considered in the last section with transition rates (9.10) and equilibrium distribution (9.12). The process considered there imposed a restriction on the parameters $\lambda(n, m)$, but it allowed the transition rates to involve the function $\psi(m)$. The relationship between the two processes is analogous to that between the migration processes of Section 2.3 and the reversible migration processes of Chapter 6.

We shall now show that the closed migration process of Section 2.3 can itself be viewed as a spatial process provided the graph G is taken as the complete graph in which every pair of sites is joined by an edge. Let $\mathcal{N}_j = \{1, 2, \ldots, N\}$ and $\lambda_j(n, m) = \lambda(n, m)$. Let $M(i)$ be the number of sites with attribute i. Thus $\mathbf{M} = (M(1), M(2), \ldots, M(N))$ is a function of $\mathbf{n}$. Let

$$\Phi(\mathbf{n}) = \prod_{i=1}^{N} \frac{1}{M(i)!} \prod_{r=1}^{M(i)} \phi_i(r)$$

and

$$\Phi_{G-j}(\mathbf{n}_{G-j}) = \frac{M(n_j)\Phi(\mathbf{n})}{\phi_{n_j}(M(n_j))}$$

Observe that Φ_{G-j} is indeed a function of $\mathbf{n}_{G-j}$. Substituting these functions into equation (9.15) gives the transition rates of the process $\mathbf{n}$ as

$$q(\mathbf{n}, T_j^m \mathbf{n}) = \lambda(n_j, m) \frac{\phi_{n_j}(M(n_j))}{M(n_j)}$$

The process $\mathbf{M}$ is also Markov and its transition rates take a simpler form. Using the operator T_{ik} introduced in Section 2.3 the process $\mathbf{M}$ has transition rates

$$q(\mathbf{M}, T_{ik}\mathbf{M}) = \lambda(i, k)\phi_i(M(i))$$

It is thus a closed migration process of the form discussed in Section 2.3. The process $\mathbf{M}$ can be viewed as a summary of the information contained in the process $\mathbf{n}$: n_j records the colony which contains individual j and $M(i)$ records the number of individuals in colony i. The transition rates and equilibrium distribution of the process $\mathbf{M}$ take the more natural form. On the other hand, the process $\mathbf{n}$ has the advantage that as a spatial process only one of its components can change at a time; in the next section we shall see that this facilitates a discussion of partial balance. The partial balance equations (2.5) for the closed migration process $\mathbf{M}$ can be deduced from the partial balance equations (9.18) for the spatial process $\mathbf{n}$; the probability flux out of state $\mathbf{M}$ due to an individual moving from colony i is equal to the probability flux into state $\mathbf{M}$ due to an individual moving to colony i, since the probability flux out of state $\mathbf{n}$ due to individual j moving from colony i is equal to the probability flux into state $\mathbf{n}$ due to individual j moving to colony i.

Let us return now to the general process with transition rates (9.15). Consider the period of time for which a site's attribute remains unchanged. We can imagine that after the attribute of site j becomes n_j the attribute has a nominal lifetime exponentially distributed with unit mean which it ages through at rate

$$\lambda_j(n_j)\frac{\Phi(\mathbf{n})}{\Phi_{G-j}(\mathbf{n}_{G-j})}$$

where

$$\lambda_j(n_j)=\sum_m \lambda_j(n_j, m)$$

and that when the attribute's lifetime ends site j takes on attribute m with probability $\lambda_j(n_j, m)/\lambda_j(n_j)$. It is clear that this description of the evolution of the system is consistent with the transition rates (9.15). Now suppose that an attribute's nominal lifetime has some arbitrary distribution with unit mean, where this distribution may vary from attribute to attribute and from site to site. The process $\mathbf{n}(t)$ will no longer be a Markov process, but our experience with migration processes suggests that its equilibrium distribution may still be given by expression (9.17). We shall now give a brief indication of how this can be proved; in the next section we shall see it is a consequence of a more general result. Suppose, to begin with, that all nominal lifetimes are exponentially distributed apart from one attribute at one site. Suppose that at site 1 attribute 1 has as a nominal lifetime the sum of w independent stages, each exponentially distributed with mean w^{-1}. Consider the process $\mathbf{n}'=(n_1', n_2, n_3, \ldots, n_J)$ with $n_1'=n_1$ when $n_1\neq 1$, and with $n_1'=(1, u)$ when $n_1=1$, where the indicator u takes a value between 1 and w depending on which stage of the attribute's lifetime is in progress. Although the process $\mathbf{n}(t)$ is not Markov its value can be deduced from the process $\mathbf{n}'(t)$ which *is* Markov. The transition rates of the process $\mathbf{n}'(t)$ are

$$q(\mathbf{n}', T_j^m\mathbf{n}')=\lambda_j(n_j, m)\frac{\Phi(\mathbf{n})}{\Phi_{G-j}(\mathbf{n}_{G-j})} \tag{9.23}$$

unless $j=1$, in which case the transition rate must be defined more carefully. If $n_1\neq 1$ then the rate at which n_1 changes to $(1, 1)$ is

$$\lambda_1(n_1, 1)\frac{\Phi(\mathbf{n})}{\Phi_{G-j}(\mathbf{n}_{G-j})}$$

If $n_1'=(1, u)$ for $1\leq u\leq w-1$ then the rate at which n_1' changes to $(1, u+1)$ is

$$\lambda_1(1)w\frac{\Phi(\mathbf{n})}{\Phi_{G-j}(\mathbf{n}_{G-j})}$$

If $n_1' = (1, w)$ then the rate at which n_1' changes to m is

$$\lambda_1(1, m)w \frac{\Phi(\mathbf{n})}{\Phi_{G-j}(\mathbf{n}_{G-j})}$$

All other transitions involving site 1 have their rates given by expression (9.23). Thus Theorem 9.4 applies to the process $\mathbf{n}'$, and from this it can be deduced that the equilibrium distribution is

$$\pi'(\mathbf{n}') = B \frac{\alpha_1'(n_1') \prod_{j=2}^{J} \alpha_j(n_j)}{\Phi(\mathbf{n})} \tag{9.24}$$

where

$$\alpha_1'(n_1') = \alpha_1(n_1) \qquad n_1 \neq 1$$

$$\alpha_1'(1, u) = \frac{1}{w} \alpha_1(1)$$

All that needs to be checked is that if α_1 is a solution of equations (9.16) then α_1' is the appropriate solution for the process $\mathbf{n}'$. But now the equilibrium distribution for $\mathbf{n}$ can be obtained by a simple summation of the distribution (9.24). This shows that

$$\pi(\mathbf{n}) = B \frac{\prod_{j=1}^{J} \alpha_j(n_j)}{\Phi(\mathbf{n})}$$

and so we have established the desired result in the case where one attribute has a nominal lifetime with a gamma distribution. At the cost of some additional notation the result can be established when nominal lifetimes of any number of attributes are distributed as mixtures of gamma distributions. As in Section 3.3 this strongly suggests the result for arbitrary distributions, but again the techniques needed for this step are beyond the scope of this work.

To illustrate the result, consider the process with transition rates (9.21) where $n_j = 0, 1$, $\lambda(0, 1) = \lambda$, $\lambda(1, 0) = \mu$, $\phi(0) = 1$, and $\phi(1) = \phi$. Interpret 0 or 1 as indicating the absence or presence of a plant at a site. Thus sites remain vacant for periods of time which have mean λ^{-1}. The nominal lifetime is equal to the actual lifetime for a vacant period. Although the vacant period can be arbitrarily distributed it may be reasonable to suppose that it has an exponential distribution if, for example, plants appear through seeds settling at random from the atmosphere. After a plant appears it ages through its nominal lifetime, arbitrarily distributed with unit mean, at rate $\mu\phi^r$ while r of the neighbouring sites have plants at them. Depending on whether ϕ is greater or less than unity a plant shortens or lengthens the actual lifetime of its neighbours. The equilibrium distribution is

$$\pi(\mathbf{n}) = B \left(\frac{\lambda}{\mu}\right)^{\Sigma n_j} \phi^{-R} \tag{9.25}$$

where R is the number of neighbouring pairs of plants, and this distribution is insensitive to the form of the nominal lifetime distributions in the model.

Exercises 9.3

1. If the process $\mathbf{n}(t)$ has transition rates (9.15) show that the reversed process $\mathbf{n}(-t)$ has transition rates of the same form, but with $\lambda_j(n_j, m)$ replaced by $\alpha_j(m)\lambda_j(m, n_j)/\alpha_j(n_j)$.
2. Show that a closed reversible migration process $\mathbf{M}$, of the form introduced in Section 6.1, can be viewed as a reversible spatial process $\mathbf{n}$ in which

$$q(\mathbf{n}, T_j^m\mathbf{n}) = \lambda(n_j, m)\frac{\phi_{n_j}(M(n_j))}{M(n_j)}\psi_{n_m}(M(n_m))$$

   where the relationship between $\mathbf{M}$ and $\mathbf{n}$ is as in the preceding section. Observe that the process $\mathbf{n}$ is a reversible spatial process but does not have transition rates of the form (9.15).
3. Observe that the plant infection model leading to the equilibrium distribution (9.14) has transition rates of the form (9.15), with

$$\Phi(\mathbf{n}) = (1+\delta)^{-R(1)}$$

   and

$$\Phi_{G-j}(\mathbf{n}_{G-j}) = \Phi(T_j^1\mathbf{n}).$$

   Deduce that the equilibrium distribution (9.14) remains valid even when the duration of infection is arbitrarily distributed. Do the same for the plant infection model of Exercise 9.2.3. Observe that the process $\mathbf{n}(t)$, while not Markov, is reversible.
4. Show that the equilibrium distribution obtained for the power supply model of Exercise 9.2.4 is of the same form even when the period for which a source remains broken down is arbitrarily distributed.
5. Many of the models discussed in earlier chapters can readily be converted into spatial processes. Consider, for example, the model of a switching system described in Section 4.4. Show that if each of the K_1+K_2 lines is regarded as a site this model becomes a special case of the process described in this section.
6. A criticism of the plant birth and death model leading to the distribution (9.25) is that it is unlikely that the plants will be constrained to exist at a finite number of sites. Consider then the following model. Suppose that the points in time at which plants are born form a Poisson process and that when a plant is born its position is chosen at random from a uniform distribution over a fixed bounded region of the plane. Suppose, further, that a plant ages through its nominal lifetime at rate $\mu\phi^r$ where r is the number of plants within a distance d of it and $\phi \geq 1$. Approximate the process by the model of the preceding section, with the graph G taken as

a very fine grid of points covering the region and with two points of this grid defined as neighbours if they are within a distance d of each other. Use the approximation to show that conditional on there being N plants alive at a given point in time the probability they take up a given configuration in the region depends on that configuration only through R, the number of pairs of plants within a distance d of each other in the configuration. When $\phi < 1$ the process does not have an equilibrium distribution; the expected number of plants in existence grows without limit. This difficulty was avoided, of course, when plants could exist at only a finite number of sites.

7. Elaborate migration processes can be constructed which have equilibrium distributions similar to those encountered in this chapter. In this exercise and the next we give some examples. Suppose the process $\mathbf{n}(t)$ has the following transition rates:

$$q(\mathbf{n}, T_{jk}\mathbf{n}) = \lambda_{jk}\frac{\Phi(\mathbf{n})}{\Phi(T_{j\cdot}\mathbf{n})}$$

$$q(\mathbf{n}, T_{j\cdot}\mathbf{n}) = \mu_j\frac{\Phi(\mathbf{n})}{\Phi(T_{j\cdot}\mathbf{n})}$$

$$q(\mathbf{n}, T_{\cdot k}\mathbf{n}) = \nu_k$$

Assuming an equilibrium distribution exists, show that it takes the form

$$\pi(\mathbf{n}) = B\frac{\prod_{j=1}^{J}\alpha_j^{n_j}}{\Phi(\mathbf{n})}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_J$ is the solution of equations (2.9). Observe that we obtain the open migration process of Chapter 2 with

$$\Phi(\mathbf{n}) = \prod_{j=1}^{J}\prod_{r=1}^{n_j}\phi_j(r)$$

Observe also that the same procedure as used in the preceding section establishes that an individual's nominal lifetime in a colony can be arbitrarily distributed without affecting the equilibrium distribution $\pi(\mathbf{n})$.

8. Consider the following process in which each site can contain at most one particle. Let

$$q(\mathbf{n}, T_{jk}\mathbf{n}) = \lambda_{jk}\phi^r$$

$$q(\mathbf{n}, T_{j\cdot}\mathbf{n}) = \mu_j\phi^r$$

$$q(\mathbf{n}, T_{\cdot k}\mathbf{n}) = \nu_k$$

for j, k such that $n_j = 1$, $n_k = 0$, where r is the number of particles occupying sites neighbouring site j. Show that if $\alpha_1, \alpha_2, \ldots, \alpha_J$ satisfy equations (6.4) and (6.7) then the equilibrium distribution for the process is

$$\pi(\mathbf{n}) = B\phi^{-R} \prod_{j=1}^{J} \alpha_j^{n_j}$$

where R is the number of edges of the graph G which have occupied sites at both ends. Observe that $\pi(\mathbf{n})$ is a Markov field.

9.4 PARTIAL BALANCE

We have come across partial balance equations frequently in this work. In Chapter 3 we saw that quasi-reversibility was equivalent to a particular form of partial balance, and we have often found that models displaying partial balance possess an insensitivity property. In this section we shall investigate further the relationships between reversed processes, partial balance, and the phenomenon of insensitivity within the relatively simple setting provided by this chapter's definition of a spatial process.

The concept of partial balance was introduced in Chapter 1, where it was shown that some of the properties of a reversible process could be obtained from an assumption weaker than detailed balance. The following theorem summarizes Exercises 1.6.2, 1.6.3, 1.6.4, 1.7.7, and 1.7.8.

Theorem 9.5. *For a Markov process with transition rates $q(j, k)$, $j, k \in \mathcal{S}$, and equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, the following statements are equivalent:*

(i) *The distribution $\pi(j)$, $j \in \mathcal{S}$, satisfies the partial balance conditions*

$$\pi(j) \sum_{k \in \mathcal{A}} q(j, k) = \sum_{k \in \mathcal{A}} \pi(k) q(k, j) \qquad j \in \mathcal{A}$$

(ii) *If the process is truncated to the set $\mathcal{A}$ the equilibrium distribution of the truncated process is the conditional probability distribution*

$$\frac{\pi(j)}{\sum_{k \in \mathcal{A}} \pi(k)} \qquad j \in \mathcal{A}$$

(iii) *If the process is altered by changing the transition rate $q(j, k)$ to $cq(j, k)$ for $j, k \in \mathcal{A}$, where $c \neq 0$ or 1, then the resulting process has the unaltered equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$.*

(iv) *If the process is altered by changing the transition rate $q(j, k)$ to $cq(j, k)$ for $j \in \mathcal{A}$, $k \in \mathcal{S} - \mathcal{A}$, where $c \neq 0$ or 1, then the equilibrium distribution of the resulting process takes the form*

$$\begin{array}{ll} B\pi(j) & j \in \mathcal{A} \\ Bc\pi(j) & j \in \mathcal{S} - \mathcal{A} \end{array}$$

(v) *The operations of time reversal and truncation to the set $\mathcal{A}$ commute.*

If in addition

$$\sum_{j\in\mathcal{A}} \sum_{k\in\mathcal{S}-\mathcal{A}} \pi(j)q(j, k)$$

is finite, then statements (i) *to* (v) *are equivalent to:*

(vi) *The Markov chain formed by observing the process at those instants in time just before it leaves the set $\mathcal{A}$ has the same equilibrium distribution as the Markov chain formed by observing the process at those instants in time just after it enters the set $\mathcal{A}$.*

When the Markov process has some additional structure the above results can usually be reformulated to make use of that structure. If the Markov process is a spatial process there are various truncations of the state space which take an intuitively appealing form. For example, single out a particular site j and suppose that the attributes of the other sites are held fixed at $\mathbf{n}_{G-j}$ with transitions involving changes at these sites forbidden. Under this truncation of the state space the attribute of site j, n_j, becomes a Markov process; the transition rate from n_j to m is $q(\mathbf{n}, T_j^m\mathbf{n})$ and thus depends on the frozen state of the rest of the system, $\mathbf{n}_{G-j}$, or at least on that part of it, $\mathbf{n}_{\partial j}$. Let $\pi(n_j; \mathbf{n}_{G-j})$, $n_j \in \mathcal{N}_j$, be the equilibrium distribution for this Markov process. Observe that a different truncation of the state space, and hence a different set of associated partial balance conditions, results from each choice of $\mathbf{n}_{G-j}$. Grouping these sets of partial balance conditions together we can obtain the following result.

Corollary 9.6. *For a spatial process with equilibrium distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$, the following statements are equivalent:*

(i) *The distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$, satisfies the partial balance equations*

$$\pi(\mathbf{n}) \sum_m q(\mathbf{n}, T_j^m\mathbf{n}) = \sum_m \pi(T_j^m\mathbf{n})q(T_j^m\mathbf{n}, \mathbf{n}) \qquad \mathbf{n} \in \mathcal{S} \qquad (9.26)$$

(ii) *The equilibrium distribution for the truncated process n_j obtained when sites other than j are frozen at $\mathbf{n}_{G-j}$ satisfies*

$$\pi(n_j; \mathbf{n}_{G-j}) = P(n_j \mid \mathbf{n}_{G-j}) \qquad \mathbf{n} \in \mathcal{S}$$

where $P(n_j \mid \mathbf{n}_{G-j})$ is the conditional probability distribution (9.1).

(iii) *If the process is altered by changing the transition rate $q(\mathbf{n}, T_j^m\mathbf{n})$ to $cq(\mathbf{n}, T_j^m\mathbf{n})$ for $\mathbf{n} \in \mathcal{S}$, $m \in \mathcal{N}_j$, where $c \neq 0$ or 1, then the resulting process has the unaltered equilibrium distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$.*

(iv) *The operations of reversing time and freezing sites other than j at $\mathbf{n}_{G-j}$ commute, for all values of $\mathbf{n}_{G-j}$.*

The alteration to the process proposed in statement (ii) corresponds to a speeding up or slowing down of transitions altering the attribute of site j. Since the equilibrium distribution is unaffected by the value of the constant c the statement can be strengthened to allow a time-varying function $c(t)$. Indeed, a version of the statement can be formulated in which $c(t)$ is itself a stochastic process.

Since the transition rates $q(\mathbf{n}, T_j^m\mathbf{n})$ of the truncated process n_j do not depend on $\mathbf{n}_{G-j-\partial j}$ its equilibrium distribution $\pi(n_j; \mathbf{n}_{G-j})$ cannot depend on $\mathbf{n}_{G-j-\partial j}$ either. Hence from statement (ii) of Corollary 9.6 we obtain the following strengthening of Theorem 9.3.

Corollary 9.7. *If the equilibrium distribution of a spatial process satisfies the partial balance equations (9.26) for each $j \in G$ then it is a Markov field.*

From now on we shall not be much concerned with the spatial aspects of the process $\mathbf{n}(t)$ and the graph G may as well be the complete graph in which every pair of sites is connected by an edge. In this case the main feature of a spatial process $\mathbf{n}(t)$ is that only one of its components can change at a time.

Rather than freezing all sites except one suppose now that just one site is frozen. In particular suppose that the attribute of site j is frozen at n_j.

Corollary 9.8. *For a spatial process with equilibrium distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$, the following statements are equivalent:*

(i) *The distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$, satisfies the partial balance equations*

$$\pi(\mathbf{n}) \sum_m q(\mathbf{n}, T_j^m\mathbf{n}) = \sum_m \pi(T_j^m\mathbf{n}) q(T_j^m\mathbf{n}, \mathbf{n}) \tag{9.27}$$

for the particular attribute n_j and for all values of $\mathbf{n}_{G-j}$.

(ii) *The equilibrium distribution for the truncated process $\mathbf{n}_{G-j}$ obtained when site j is frozen at n_j is the conditional probability distribution $P(\mathbf{n}_{G-j} \mid n_j)$ induced by the equilibrium distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$.*

(iii) *If the process is altered by changing the transition rates $q(\mathbf{n}, T_j^m\mathbf{n})$ to $cq(\mathbf{n}, T_j^m\mathbf{n})$ for the particular attribute n_j and for all values of $\mathbf{n}_{G-j}$, m, where $c \neq 0$ or 1, then the equilibrium probability that the resulting process is in state $T_j^m\mathbf{n}$ takes the form*

$$\begin{array}{ll} B\pi(\mathbf{n}) & m = n_j \\ Bc\pi(T_j^m\mathbf{n}) & \text{otherwise} \end{array}$$

(iv) *The operations of reversing time and freezing site j at n_j commute.*

(v) *The equilibrium probability that the process is in state $\mathbf{n}$ given that the attribute of site j has just become n_j is the same as the equilibrium probability that the process is in state $\mathbf{n}$ given that the attribute of site j is about to change from n_j, for all values of $\mathbf{n}_{G-j}$.*

Proof. Let the set $\mathscr{A}$ be all states in $\mathscr{S}$ in which site j has the particular attribute n_j. Statement (i) is obtained from statement (i) of Theorem 9.5 by subtracting the partial balance conditions from the equilibrium equations. Statements (ii), (iii), and (iv) are just statements (ii), (iv), and (v) of Theorem 9.5. The probability flux out of the set $\mathscr{A}$ is finite since $\mathscr{S}$, and hence $\mathscr{A}$, is assumed to be finite. Thus statement (vi) of Theorem 9.5 applies, giving statement (v) of the present corollary.

Observe that the alteration to the process described in statement (iii) corresponds to a speeding up or slowing down of transitions ending the particular attribute n_j.

Of course freezing all sites other than one is not in principle very different from freezing just one site. Indeed, if we regard all sites other than one as forming a composite site the operations are identical. The important difference between Corollary 9.6 and Corollary 9.8 is that the partial balance equations (9.27) concern a particular attribute n_j while the partial balance equations (9.26) allow n_j to range over the set $\mathcal{N}_j$.

The consequences of partial balance described above are reminiscent of results obtained by other means in earlier chapters. There is, for example, a close relationship between statement (ii) of Theorem 9.5 and part (iii) of Theorem 3.12, or between statement (iv) of Theorem 9.5 and the product form obtained in Section 2.3. Statement (iv) of Corollary 9.6 gives an insight into why the reversed process obtained from a migration process takes such a simple form. These correspondences can be made precise by appropriately reformulating the models of earlier chapters, but this approach is generally too cumbersome to be useful.

We have not yet discussed the relationship between partial balance and the phenomenon of insensitivity, but the equivalence between statements (i) and (ii) of Corollary 9.6 suggests the following very rough line of argument. If the nominal lifetimes of the attributes at site j are arbitrarily rather than exponentially distributed then this will not affect the equilibrium distribution of the truncated process n_j. Hence it should not affect the equilibrium distribution $\pi(\mathbf{n})$ of the process $\mathbf{n}(t)$. We shall now make this argument more precise.

Let $\mathbf{x}=(x_1, x_2, \ldots, x_J)$ be a spatial process with state space $\mathscr{X}_1 \times \mathscr{X}_2 \times \cdots \times \mathscr{X}_J$, and let $\pi(x_j; \mathbf{x}_{G-j})$, $x_j \in \mathscr{X}_j$, be the equilibrium distribution for the truncated process x_j obtained when sites other than j are frozen at $\mathbf{x}_{G-j}$. Thus

$$\pi(x_j; \mathbf{x}_{G-j}) \sum_y q(\mathbf{x}, T_j^y \mathbf{x}) = \sum_y \pi(y; \mathbf{x}_{G-j}) q(T_j^y \mathbf{x}, \mathbf{x}) \tag{9.28}$$

Suppose that for each $j \in G$ there is a function

$$f_j: \mathscr{X}_j \to \mathcal{N}_j$$

Let $n_j = f_j(x_j)$ and write $\mathbf{n} = (n_1, n_2, \ldots, n_J)$ for $f(\mathbf{x}) = (f_1(x_1), f_2(x_2), \ldots, f_J(x_J))$. Thus n_j is a less detailed description than x_j of the attribute of site j. We can imagine that the dependence between site j and sites other than j is through n_j rather than x_j. To formalize this idea we shall make two assumptions about the truncated process x_j. The first assumption is that the transition rates $q(\mathbf{x}, T_j^y\mathbf{x})$ depend on $\mathbf{x}_{G-j}$ only through $\mathbf{n}_{G-j}$. Thus $\pi(x_j; \mathbf{x}_{G-j})$ can depend on $\mathbf{x}_{G-j}$ only through $\mathbf{n}_{G-j}$ and so can be written $\pi(x_j; \mathbf{n}_{G-j})$. Let

$$\pi(n_j; \mathbf{n}_{G-j}) = \sum_{x_j : f_j(x_j) = n_j} \pi(x_j; \mathbf{n}_{G-j})$$

The second assumption is that $\pi(x_j; \mathbf{n}_{G-j})$ can be written in the form

$$\pi(x_j; \mathbf{n}_{G-j}) = \pi(n_j; \mathbf{n}_{G-j}) P_j(x_j \mid n_j) \tag{9.29}$$

This assumption is best regarded as a condition on the function f_j; put in statistical terms relation (9.29) asserts that the statistic $n_j = f_j(x_j)$ obtained from the data x_j is sufficient for the parameter $\mathbf{n}_{G-j}$. Put another way the assumption is that conditional on n_j the attribute x_j has a distribution $P_j(x_j \mid n_j)$ which does not depend on $\mathbf{n}_{G-j}$. Both assumptions together allow us to write

$$\pi(x_j; \mathbf{x}_{G-j}) = \pi(n_j; \mathbf{n}_{G-j}) P_j(x_j \mid n_j) \tag{9.30}$$

where $\mathbf{n} = f(\mathbf{x})$.

If we had to choose transition rates for a spatial process so that it resembled the process $\mathbf{n} = f(\mathbf{x})$ how would we do it? One way would be to set

$$q(\mathbf{n}, T_j^m\mathbf{n}) = \sum_{x_j : f_j(x_j) = n_j} P_j(x_j \mid n_j) \sum_{y : f_j(y) = m} q(\mathbf{x}, T_j^y\mathbf{x}) \tag{9.31}$$

Observe that if the truncated process x_j is in equilibrium the right-hand side of equation (9.31) is the probability intensity that $f_j(x_j)$ changes to m given that it starts equal to n_j.

Theorem 9.9. *Suppose a spatial process $\mathbf{x}(t)$ is such that the truncated process x_j satisfies both the above assumptions for each $j \in G$. Let the rates $q(\mathbf{n}, T_j^m\mathbf{n})$ be defined by equation (9.31). If there exists a distribution $\pi(\mathbf{n})$ satisfying the partial balance equations (9.26) for each $j \in G$ then the equilibrium distribution for the process $\mathbf{x}(t)$ is*

$$\pi(\mathbf{x}) = \pi(\mathbf{n}) \prod_{j=1}^{J} P_j(x_j \mid n_j) \tag{9.32}$$

and it satisfies the partial balance equations

$$\pi(\mathbf{x}) \sum_{y \in \mathscr{X}_j} q(\mathbf{x}, T_j^y\mathbf{x}) = \sum_{y \in \mathscr{X}_j} \pi(T_j^y\mathbf{x}) q(T_j^y\mathbf{x}, \mathbf{x})$$

for each $j \in G$ and for all $\mathbf{x}$.

Proof. Substituting the form (9.30) into equations (9.28) we obtain

$$\pi(n_j; \mathbf{n}_{G-j}) P_j(x_j \mid n_j) \sum_{m \in \mathcal{N}_j} \sum_{y: f_j(y)=m} q(\mathbf{x}, T_j^y \mathbf{x})$$
$$= \sum_{m \in \mathcal{N}_j} \sum_{y: f_j(y)=m} \pi(m; \mathbf{n}_{G-j}) P_j(y \mid m) q(T_j^y \mathbf{x}, \mathbf{x}) \tag{9.33}$$

If these equations are summed over x_j such $f_j(x_j) = n_j$ they reduce to

$$\pi(n_j; \mathbf{n}_{G-j}) \sum_m q(\mathbf{n}, T_j^m \mathbf{n}) = \sum_m \pi(m; \mathbf{n}_{G-j}) q(T_j^m \mathbf{n}, \mathbf{n}) \tag{9.34}$$

using equation (9.31). Since $\pi(\mathbf{n})$ satisfies the partial balance equations the distribution satisfying equations (9.34) is

$$\pi(n_j; \mathbf{n}_{G-j}) = \frac{\pi(\mathbf{n})}{\sum_{m \in \mathcal{N}_j} \pi(T_j^m \mathbf{n})}$$

Substituting this back into equations (9.33) confirms that the proposed equilibrium distribution $\pi(\mathbf{x})$ satisfies the partial balance equations for the process $\mathbf{x}(t)$. Hence $\pi(\mathbf{x})$ is indeed the equilibrium distribution for the process $\mathbf{x}(t)$.

Consider now a spatial process $\mathbf{n}$ with transition rates $q(\mathbf{n}, T_j^m \mathbf{n})$. Consider in particular the period of time for which a site's attribute remains unchanged. We can imagine that after the attribute of site j becomes n_j the attribute has a nominal lifetime exponentially distributed with unit mean which it ages through at rate

$$\sum_m q(\mathbf{n}, T_j^m \mathbf{n}) \tag{9.35}$$

and that when the attribute's lifetime ends site j takes on attribute n with probability

$$\frac{q(\mathbf{n}, T_j^n \mathbf{n})}{\sum_m q(\mathbf{n}, T_j^m \mathbf{n})} \tag{9.36}$$

Clearly this description of the evolution of the system is consistent with the transition rates $q(\mathbf{n}, T_j^m \mathbf{n})$. Now suppose that an attribute's nominal lifetime has some arbitrary distribution with unit mean, where this distribution may vary from attribute to attribute and from site to site. The process $\mathbf{n}(t)$ will no longer be a Markov process, but if the partial balance equations (9.26) are satisfied its equilibrium distribution will still be given by $\pi(\mathbf{n})$. We shall establish this fact for the case where nominal lifetimes are distributed as mixtures of gamma distributions using the method of stages.

Suppose that when site j takes on attribute m it in fact takes on a finer attribute, (m, z), with probability $p_j(m, z)$, where z belongs to a countable set

$\mathscr{Z}$ and $\sum_z p_j(m, z) = 1$ for each j and z. Suppose that the nominal lifetime of the finer attribute is made up of $w_j(m, z)$ independent stages, each exponentially distributed with mean $d_j(m, z)$. This framework allows the nominal lifetime of attribute m at site j to have any distribution which can be expressed as a mixture of gamma distributions. Set

$$\sum_z p_j(m, z) w_j(m, z) d_j(m, z) = 1 \tag{9.37}$$

so that nominal lifetimes have unit mean. Use $x_j = (n_j, z_j, u_j)$ to describe site j where (n_j, z_j) is the finer attribute of the site and $u_j(1 \leq u_j \leq w_j(n_j, z_j))$ is the stage currently in progress. Observe that the description $\mathbf{x} = (x_1, x_2, \ldots, x_J)$ is detailed enough to be a Markov process. Consider now the reduction $\mathbf{n} = f(\mathbf{x})$ given by $n_j = f_j(x_j)$, $j \in G$. The transition rates of the truncated process x_j depend on $\mathbf{x}_{G-j}$ only through $\mathbf{n}_{G-j}$, and hence we can write

$$\pi(x_j; \mathbf{x}_{G-j}) = \pi(x_j; \mathbf{n}_{G-j})$$

Considering the truncated process x_j more closely it is apparent that although $\mathbf{n}_{G-j}$ affects the *rate* at which transition occur it does not affect *which* transitions occur except possibly when n_j changes. If we observe the truncated process x_j only while n_j takes the value m the proportion of time for which (z_j, u_j) takes the value (z, u) is $p_j(m, z)d_j(m, z)$ and is unaffected by $\mathbf{n}_{G-j}$. Hence we can write

$$\pi(x_j; \mathbf{n}_{G-j}) = \pi(n_j; \mathbf{n}_{G-j}) P_j(x_j \mid n_j)$$

where

$$P_j(x_j \mid n_j) = P_j(z_j, u_j \mid n_j) = p_j(n_j, z_j) d_j(n_j, z_j) \qquad z_j \in \mathscr{Z}, \quad 1 \leq u_j \leq w_j(n_j, z_j)$$

Equation (9.37) ensures that the distribution $P_j(x_j \mid n_j)$ sums to unity. It is readily checked that the rates $q(\mathbf{n}, T_j^m \mathbf{n})$ are consistent with the definition (9.31). We can thus use Theorem 9.9 to deduce that if $\pi(\mathbf{n})$ satisfies the partial balance equations (9.26) then the equilibrium distribution for the process $\mathbf{x}$ takes the form (9.32). Hence the process $\mathbf{n}$ has the equilibrium distribution $\pi(\mathbf{n})$ and is insensitive to the exact form of the nominal lifetime distributions provided they arise as mixtures of gamma distributions. The restriction to mixtures of gamma distributions can be removed, but we shall not discuss this.

There are stronger forms of insensitivity than that just described, and we shall briefly describe one. Suppose the transition rates $q(\mathbf{n}, T_j^m \mathbf{n})$ allow a solution to the partial balance equations (9.26) and further suppose the ratio (9.36) does not depend upon $\mathbf{n}_{G-j}$, as, for example, in the general spatial process described in the last section. We can then allow the next attribute to be taken on by site j and that attribute's nominal lifetime to depend upon the previous sequence of attributes taken by site j and upon their nominal lifetimes: if the process is stationary the equilibrium distribution for $\mathbf{n}$ will be insensitive to these dependencies. The reader will observe the parallel with the pattern of dependencies allowed in closed queueing networks. We shall not prove the result, but the idea is to extend the description x_j to include a summary of the history of site j. The condition on the ratio (9.36) ensures that the jump chain of the truncated process x_j is independent of $\mathbf{n}_{G-j}$, and this in turn shows that $\mathbf{n}_{G-j}$ and x_j are independent, conditional on n_j. The result is not really surprising in view of Corollary 9.6; the equilibrium distribution for the truncated process n_j will be determined by the proportion of time for which site j takes each attribute, no matter how complicated the pattern of dependencies.

In the above discussion it was assumed that the equilibrium distribution of the spatial process satisfied all the partial balance equations (9.26). If the equilibrium distribution satisfies the smaller set of partial balance equations described in statement (i) of Corollary 9.8 then it is possible to show that the nominal lifetime of the particular attribute n_j can be arbitrarily distributed with unit mean without it affecting the equilibrium distribution $\pi(\mathbf{n})$ (Exercise 9.4.1); this might be intuitively expected in view of the equivalence between statements (i) and (ii) of Corollary 9.8. We shall now prove a converse to this result.

Theorem 9.10. *If in a spatial process the nominal lifetime of a particular attribute, n_j, can have any distribution with unit mean without it affecting the equilibrium distribution $\pi(\mathbf{n})$, $\mathbf{n}\in\mathcal{S}$, then*

$$\pi(\mathbf{n})\sum_m q(\mathbf{n}, T_j^m\mathbf{n})=\sum_m \pi(T_j^m\mathbf{n})q(T_j^m\mathbf{n}, \mathbf{n}) \tag{9.38}$$

for the particular attribute n_j and for all values of $\mathbf{n}_{G-j}$.

Proof. The equilibrium equations for the case where the nominal lifetime of attribute n_j is exponentially distributed show that

$$\pi(\mathbf{n})\sum_i\sum_m q(\mathbf{n}, T_i^m\mathbf{n})=\sum_i\sum_m \pi(T_i^m\mathbf{n})q(T_i^m\mathbf{n}, \mathbf{n}) \tag{9.39}$$

Now let the nominal lifetime of attribute n_j be a mixture of two exponential random variables. Suppose that when site j takes on attribute n_j the nominal lifetime of the attribute is exponentially distributed with mean either a_1 or a_2, each possibility being equally likely. To ensure that the mean nominal lifetime is unity, set $a_1+a_2=2$. The process $\mathbf{n}$ can be rendered Markov if when site j has attribute n_j we append an indication of which exponential random variable has been chosen for the attribute's nominal lifetime. Write $\pi(\mathbf{n}, 1)$, $\pi(\mathbf{n}, 2)$, $\pi(T_j^m\mathbf{n})$ for the equilibrium distribution of the resulting process. The equilibrium equations for this process show that

$$\pi(\mathbf{n},1)\left(a_1^{-1}\sum_m q(\mathbf{n},T_j^m\mathbf{n})+\sum_{i\neq j}\sum_m q(\mathbf{n},T_i^m\mathbf{n})\right)$$
$$=\tfrac{1}{2}\sum_m \pi(T_j^m\mathbf{n})q(T_j^m\mathbf{n},\mathbf{n})+\sum_{i\neq j}\sum_m \pi(T_i^m\mathbf{n},1)q(T_i^m\mathbf{n},\mathbf{n}) \qquad (9.40)$$

and

$$\pi(\mathbf{n},2)\left(a_2^{-1}\sum_m q(\mathbf{n},T_j^m\mathbf{n})+\sum_{i\neq j}\sum_m q(\mathbf{n},T_i^m\mathbf{n})\right)$$
$$=\tfrac{1}{2}\sum_m \pi(T_j^m\mathbf{n})q(T_j^m\mathbf{n},\mathbf{n})+\sum_{i\neq j}\sum_m \pi(T_i^m\mathbf{n},2)q(T_i^m\mathbf{n},\mathbf{n}) \qquad (9.41)$$

Now

$$\pi(\mathbf{n},1)+\pi(\mathbf{n},2)=\pi(\mathbf{n}) \qquad (9.42)$$

and

$$\pi(T_i^m\mathbf{n},1)+\pi(T_i^m\mathbf{n},2)=\pi(T_i^m\mathbf{n})$$

so if equations (9.40) and (9.41) are added together and the result compared with equation (9.39) we obtain

$$\pi(\mathbf{n},1)a_1^{-1}+\pi(\mathbf{n},2)a_2^{-1}=\pi(\mathbf{n}) \qquad (9.43)$$

Solving equations (9.42) and (9.43) gives

$$\pi(\mathbf{n},1)=\tfrac{1}{2}a_1\pi(\mathbf{n})$$
$$\pi(\mathbf{n},2)=\tfrac{1}{2}a_2\pi(\mathbf{n})$$

Substituting for $\pi(\mathbf{n},1)$, $\pi(T_i^m\mathbf{n},1)$ in equation (9.40) we obtain

$$\pi(\mathbf{n})\left(\tfrac{1}{2}\sum_m q(\mathbf{n},T_j^m\mathbf{n})+a_1\sum_{i\neq j}\sum_m q(\mathbf{n},T_i^m\mathbf{n})\right)$$
$$=\tfrac{1}{2}\sum_m \pi(T_j^m\mathbf{n})q(T_j^m\mathbf{n},\mathbf{n})+a_1\sum_{i\neq j}\sum_m \pi(T_i^m\mathbf{n})q(T_i^m\mathbf{n},\mathbf{n})$$

and since this equation holds for more than one value of a_1 we can deduce the result stated.

Exercises 9.4

1. If equations (9.38) hold for the particular attribute n_j and for all values of $\mathbf{n}_{G-j}$ then the equilibrium distribution $\pi(\mathbf{n})$ is insensitive to the form of the nominal lifetime distribution of attribute n_j, even if all the partial balance equations (9.26) do not hold. Establish this either directly or by appending a marker site to the system in such a way that all the partial balance equations for the extra site hold.

2. Section 6.1 contained the assertion that the equilibrium distribution $\pi(\mathbf{n})$ of a closed reversible migration process is insensitive to the form of the nominal lifetime distributions. In fact the nominal lifetimes of an individual at successive colonies can even be dependent on each other without it affecting the equilibrium distribution $\pi(\mathbf{n})$. Establish this in the case where the process can be represented by a stationary Markov process with a countable state space. Observe that since the ratio (9.36) depends on $\mathbf{n}_{G-j}$ it is not easy to formulate a version of the process in which there exist arbitrary dependencies between the route of an individual and the nominal lifetimes along that route.
3. Consider the following model of competition between two species. Let n_i be the population size of species i, for $i=1,2$. The probability intensity that a new individual of species 1 is born is n_1+1 and that an individual of species 1 dies is $n_1 \exp(-\alpha+\gamma(n_1-1)+\delta n_2)$, $\gamma, \delta>0$. The birth and death rates for species 2 are the same, with n_1 and n_2 interchanged. Observe that the form of the birth rate prevents a species from becoming extinct and corresponds to immigration at unit rate. The probability intensity that a given individual dies is increased by a factor e^{γ} by each other existing individual of the same species and by a factor e^{δ} by each other existing individual of the other species. Show that in equilibrium

$$\pi(n_1, n_2)=B \exp\left\{\alpha(n_1+n_2)-\frac{\gamma}{2}[n_1(n_1-1)+n_2(n_2-1)]-\delta n_1 n_2\right\}$$

The most probable states (n_1, n_2) fall within regions whose position depends upon the parameters α, γ, and δ. Sketch the quadratic contours along which the function $\pi(n_1, n_2)$ takes a constant value and obtain Fig. 9.2. Show that the equilibrium distribution $\pi(n_1, n_2)$ is unaffected if individuals have arbitrarily distributed nominal lifetimes.
4. Suppose the equilibrium distribution $\pi(\mathbf{n})$, $\mathbf{n}\in\mathcal{S}$, of a spatial process satisfies the partial balance equations (9.26) for each $j\in G$. Suppose now that the process is amended so that the nominal lifetimes of attributes are allowed to be arbitrarily distributed with *any* mean. Let $\alpha_j(n_j)$ be the mean nominal lifetime of attribute n_j at site j. Use Corollary 9.8 to show that the equilibrium probability that $\mathbf{n}$ describes the state of the amended process is of the form

$$B\pi(\mathbf{n})\prod_j \alpha_j(n_j) \qquad \mathbf{n}\in\mathcal{S}$$

5. Exercise 9.4.1 and Theorem 9.10 characterize a certain form of insensitivity in terms of partial balance equations, within the framework provided by the definition of a spatial process. Use Exercise 4.6.9 to define a spatial process exhibiting an alternative form of insensitivity.
6. Statement (vi) of Theorem 9.5 provides an alternative explanation of the connection between partial balance and insensitivity. Suppose that for a

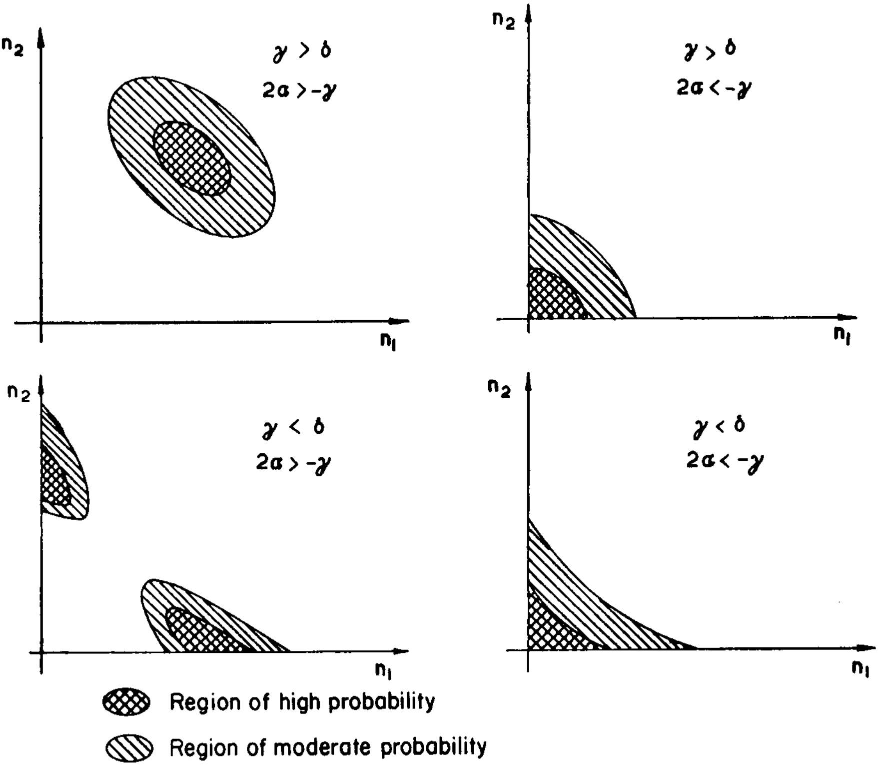

Fig. 9.2 A competition model

Markov process $X(t)$ with transition rates $q(j, k)$, $j, k \in \mathscr{S}$, and equilibrium distribution $\pi(j)$, $j \in \mathscr{S}$, statement (i) of Theorem 9.5 holds. Suppose also that $\sum_{k \in \mathscr{S}-\mathscr{A}} q(j, k)$ is the same for each $j \in \mathscr{A}$, equal to $q(\mathscr{A})$ say, and that $\mathscr{A}$ is finite; hence statement (vi) of Theorem 9.5 is true. Show that in equilibrium the distribution over states at an instant in time when the process has just entered the set $\mathscr{A}$ and the distribution over states at an instant in time when the process is just about to leave the set $\mathscr{A}$ are both identical to the equilibrium distribution of the truncated process referred to in statement (ii) of Theorem 9.5. Consider now a realization of the process $X(t)$. The process will alternate between the set $\mathscr{A}$ and the set $\mathscr{S}-\mathscr{A}$, and the periods spent in the set $\mathscr{A}$ will be exponentially distributed with parameter $q(\mathscr{A})$. Because of the identity between the three distributions referred to above the behaviour of the process during a period in the set $\mathscr{A}$ is as if it were generated by the following procedure. When the process enters the set $\mathscr{A}$ choose a random variable τ, exponentially distributed with parameter $q(\mathscr{A})$. For a period of length τ allow the process to evolve in accordance with the transition rates $q(j, k)$,

$j, k \in \mathscr{A}$, of the truncated process. At the end of this period eject the process from the set $\mathscr{A}$: if the process is in state $j \in \mathscr{A}$ move it to state $k \in \mathscr{S} - \mathscr{A}$ with probability $q(j, k)/q(\mathscr{A})$. The advantage of viewing the process in this way is that it becomes natural to allow the periods spent in the set $\mathscr{A}$ to be arbitrary rather than exponential random variables. The proportion of time the process spends in the set $\mathscr{A}$ will then be determined by the overall mean of these periods, and of the time spent in the set $\mathscr{A}$ a proportion $\pi(j)/\sum_{k \in \mathscr{A}} \pi(k)$ will be spent in state j.