CHAPTER 5
Electrical Analogues

In Section 1.4 we discussed the relationship between the approach to equilibrium of a random walk satisfying the detailed balance conditions and the diffusion of charge through an electrical network. We shall begin this chapter by discussing a different aspect of this connection between reversible random walks and electrical networks. Then, in Section 5.2, we shall use the connection to analyse a model representing flow through a network. This flow model is fairly limited in scope but its main interest lies in the fact that it permits blocking, the phenomenon whereby whether an individual leaves a colony is affected by the number of individuals in the colonies to which he may move. The structure of the migration processes and queueing networks discussed in earlier chapters ruled out blocking (although in a very limited way it could be imitated—see Exercise 4.3.4). In general the phenomenon of blocking makes analytical progress difficult; the flow model of this chapter and the reversible migration processes of the next, although highly specialized systems, at least permit blocking and yet remain tractable.

The method we shall use to analyse the flow model can also be applied to an interesting invasion model, and this will be the subject of Section 5.3.

5.1 RANDOM WALKS

Let λ_{jk} be the transition rate from state j to state k of a Markov process with a finite state space G. Let α_j, $j \in G$, be a positive solution of the equations

$$\alpha_j \sum_k \lambda_{jk} = \sum_k \alpha_k \lambda_{kj} \qquad j \in G \tag{5.1}$$

Assume as usual that the transition rates λ_{jk}, $j, k \in G$, define an irreducible Markov process, and hence that the solution to equations (5.1) is unique up to a multiplying factor. We shall regard the Markov process as defining the position of a particle performing a random walk on the graph with vertex set G and with an edge joining $j, k \in G$ if either λ_{jk} or λ_{kj} is positive. If

$$\alpha_j \lambda_{jk} = \alpha_k \lambda_{kj} \qquad j, k \in G \tag{5.2}$$

then we will call the random walk reversible. Thus in this chapter we view reversibility as a property of the transition rates of the Markov process rather than, as in earlier chapters, a property of the equilibrium behaviour of the Markov process. If

$$\lambda_{jk} = \lambda_{kj} \qquad j, k \in G$$

then we will call the random walk symmetric. A symmetric random walk allows $\alpha_j = 1$, $j \in G$, as a solution to equations (5.1) and (5.2), and hence is reversible.

Suppose that observation of the random walk is stopped when the particle first reaches a set $V \subset G$ and that a payment of v_i is received if observation is stopped when the particle arrives at vertex $i \in V$. Let p_j be the expected final payment if the particle starts from vertex j. Considering where the next step of the random walk will take the particle leads to the equation

$$p_j = \sum_k \frac{\lambda_{jk}}{\sum_l \lambda_{jl}} p_k \qquad j \in G - V$$

Thus

$$0 = \sum_k \lambda_{jk}(p_k - p_j) \qquad j \in G - V \tag{5.3}$$

and

$$p_j = v_j \qquad j \in V \tag{5.4}$$

Equations (5.3) and (5.4) have a unique solution for p_j, $j \in G$. Now suppose that the random walk is reversible and define the (possibly infinite) quantity

$$r_{jk} = (\alpha_j \lambda_{jk})^{-1}$$

Thus $r_{jk} = r_{kj}$. In this case equations (5.3) and (5.4) can be rewritten as

$$0 = \sum_k \frac{p_k - p_j}{r_{jk}} \qquad j \in G - V \tag{5.5}$$

and

$$p_j = v_j \qquad j \in V \tag{5.6}$$

But equations (5.5) and (5.6) are precisely Kirchhoff's equations for an electrical network with node set G in which we interpret p_j as the electrical potential of node j, and where nodes j and k are connected by a wire of resistance r_{jk}, and node j is held at potential v_j, $j \in V$. Equation (5.5) expresses the fact that the total current flowing into node j is zero.

The last paragraph dealt with the infinite horizon case in the sense that the particle was allowed as long as necessary to reach the set V. In this paragraph we will suppose there is a finite horizon at time T and that no payment is received if the particle has not reached the set V by time T. Let $p_j(t)$ be the expected final payment if the particle is at vertex j at time $T - t$, with a time t to go before the horizon. Considering the possible events in an interval of time δt leads to the equation

$$p_j(t+\delta t)=\sum_k \lambda_{jk}\,\delta t\, p_k(t)+\left(1-\sum_k \lambda_{jk}\,\delta t\right)p_j(t)+o(\delta t) \qquad j\in G-V$$

Hence

$$\frac{\mathrm{d}p_j(t)}{\mathrm{d}t}=\sum_k \lambda_{jk}(p_k(t)-p_j(t)) \qquad j\in G-V \tag{5.7}$$

and

$$p_j(t)=v_j \qquad j\in V \tag{5.8}$$

with the initial condition

$$p_j(0)=0 \qquad j\in G-V \tag{5.9}$$

Equations (5.7) are called the backward equations, in contrast to the forward equations (1.16). Equations (5.7), (5.8), and (5.9) have a unique solution for $p_j(t)$, $j\in G$, and

$$\lim_{t\to\infty} p_j(t)=p_j$$

where p_j, $j\in G$, is the solution to equations (5.3) and (5.4). If the random walk is reversible, equations (5.7), (5.8), and (5.9) become

$$\alpha_j\frac{\mathrm{d}p_j(t)}{\mathrm{d}t}=\sum_k \frac{p_k(t)-p_j(t)}{r_{jk}} \qquad j\in G-V \tag{5.10}$$

and

$$p_j(t)=v_j \qquad j\in V \tag{5.11}$$

with the initial condition

$$p_j(0)=0 \qquad j\in G-V \tag{5.12}$$

Equations (5.10), (5.11), and (5.12) have an electrical interpretation: if we amend the electrical network described in the last paragraph by connecting each node $j\in G-V$ to earth through a capacitor with capacitance α_j and if the potential of each node $j\in G-V$ is zero at time $t=0$, then $p_j(t)$ will be the potential of node j at time t. Observe that as far as the resistors and capacitors are concerned the electrical network is the same as the one described in Section 1.4. Note, however, that in Section 1.4 the analogy was based on the forward equations (1.16) and the correspondence was between charge and probability, both represented by the variable $u_j(t)$. Here the analogy is based on the backward equations (5.7) and the correspondence is between potential and expected final payment, both represented by the variable $p_j(t)$. Of course the expected final payment $p_j(t)$ can itself be used to represent a probability. For example if $V=\{0, 1\}$, $v_0=0$, $v_1=1$, then $p_j(t)$ will be the probability that a random walk starting at vertex j reaches vertex 1 before it reaches vertex 0 and does so before a time t has elapsed.

The analogy of this section can be extended further: we will give two examples.

Example 1. Suppose the payment received is $v_i(t)$ if the particle reaches a vertex $i \in V$ with a time t to go before the horizon. Equation (5.11) will become

$$p_j(t)=v_j(t) \qquad j \in V$$

and in the electrical analogue we will require that node $j \in V$ be maintained at the time-varying potential $v_j(t)$.

Example 2. Suppose a payment of w_j is received if the particle has not reached the set V by the horizon and is at vertex $j \in G-V$ at time T. Then the initial condition (5.12) will become

$$p_j(0)=w_j \qquad j \in G-V$$

and in the electrical analogue we will require that the potential of node $j \in G-V$ at time $t=0$ be w_j.

Exercises 5.1

1. Show that equations (5.10), (5.11), and (5.12) have an alternative electrical interpretation, related to the one given, but with resistors replaced by inductors and capacitors replaced by resistors.
2. Observe that the transition rates λ_{jk}, $j \in V$, $k \in G$, do not affect the expected final payment $p_j(t)$. Deduce that the electrical analogy of this section can still be developed if conditions (5.1) and (5.2) are replaced by the weaker condition

$$\alpha_j\lambda_{jk}=\alpha_k\lambda_{kj} \qquad j, k \in G-V$$

5.2 FLOW MODELS

Consider the following flow model. There are $J-1$ sites (or colonies) labelled $2, 3, \ldots, J$, and no site may contain more than one individual. If site j is occupied and site k is empty then with probability intensity λ_{jk} the individual at site j moves to site k. If site j is occupied then with probability intensity μ_j the individual at site j leaves the system entirely. If site k is empty then with probability intensity ν_k an individual arrives at site k from outside the system.

In this section we shall analyse this flow model under the assumption that the λ_{jk} are symmetric:

$$\lambda_{jk} = \lambda_{kj} \tag{5.13}$$

(In Section 6.3 we shall consider the model with a different restriction.) We shall suppose that at time $t = 0$ the system is empty and we shall be concerned to find $p_j(t)$, the probability that at time t site j is occupied.

Now consider the following button model. There are $J+1$ sites, labelled $0, 1, \ldots, J$ and each site contains a button. The buttons are distinguishable—we can imagine them to be of different colours. The buttons occupying sites j and k interchange positions with probability intensity λ_{jk} for $j, k = 0, 1, \ldots, J$, where

$$\left.\begin{aligned} \lambda_{0j} &= \lambda_{j0} = \mu_j \\ \lambda_{1j} &= \lambda_{j1} = \nu_j \end{aligned}\right. \qquad j = 2, 3, \ldots, J$$

and

$$\lambda_{01} = \lambda_{10} = 0$$

We see that any particular button performs a symmetric (and hence reversible) random walk around the sites of the system. Now imagine that from time $t = 0$ onwards a button leaving site 1 is painted black and a button entering site 0 is painted white. If $A(t)$ is the set of sites which contain a black button at time t then $A(t)$ behaves stochastically just as does the set of occupied sites in the flow model. Thus to find $p_j(t)$ we need only look backwards through time at the movements of the button which occupies site j at time t. These movements form a symmetric random walk starting from site j with transition intensities λ_{jk} for $j, k = 0, 1, \ldots, J$; $p_j(t)$ is equal to the probability that this random walk reaches site 1 within a time t and does so without passing through site 0. Thus $p_j(t)$ is equal to the potential at time t of node j in an electrical network constructed as follows: join nodes j and k, where $j, k = 0, 1, \ldots, J$, by a wire of resistance λ_{jk}^{-1} whenever $\lambda_{jk} > 0$, and connect nodes $2, 3, \ldots, J$ to earth through a unit capacitor; let the potentials of every node at time $t = 0$ be zero and from time $t = 0$ onwards hold nodes 0 and 1 at potentials 0 and 1 respectively. If at time $t = 0$ site j in the flow model is occupied then the electrical analogy will still hold but we will require that at time $t = 0$ the potential of node j be 1. As $t \to \infty$, $p_j(t) \to p_j$ where p_j is the equilibrium potential of node j in the network; of course, in equilibrium the potentials will be unchanged if all the capacitors are removed.

The average net flow of individuals from site j to site k at time t is

$$\begin{aligned} &\lambda_{jk} \operatorname{Prob}\{\text{site } j \text{ occupied and site } k \text{ empty at time } t\} \\ &\qquad - \lambda_{kj} \operatorname{Prob}\{\text{site } j \text{ empty and site } k \text{ occupied at time } t\} \\ &\qquad\qquad = \lambda_{jk}(p_j(t) - p_k(t)) \end{aligned}$$

which is precisely the current flowing from node j to node k at time t in the electrical network. The average flow of individuals into, and out of, the system at time t are respectively

$$\sum_j \nu_j(1-p_j(t))$$

and

$$\sum_j \mu_j p_j(t)$$

which correspond respectively to the current flowing into, and out of, the electrical network at time t.

For the flow model we have shown that $p_j(t)$, $j=2,3,\ldots,J$, satisfy a set of linear differential equations of the form (5.10), (5.11), and (5.12). Now the set of occupied sites in the flow model is a Markov process and hence forward equations could be deduced which would also form a set of linear differential equations. What have we achieved? Well, first there are only $J-1$ equations in the set we have obtained while there would be 2^{J-1} forward equations. Second, the electrical analogy gives considerable insight into the behaviour of flow models. Of course, a solution to the forward equations would give much more than just a solution for $p_j(t)$, $j=2,3,\ldots,J$; for example it would give the probability that at time t sites j and k are *both* occupied. In fact a solution to the forward equations can be built up inductively starting from a solution for $p_j(t)$, $j=2,3,\ldots,J$ (see Exercise 5.2.1). If we are interested in the equilibrium behaviour of the flow model then the solution to equations (5.5) and (5.6) will give p_j, the equilibrium probability that site j is occupied. An important feature of the flow model which puts it in sharp contrast with the open network models considered in earlier chapters is that the states of different sites are not in general independent. The solution p_j, $j=2,3,\ldots,J$, does not therefore completely determine the equilibrium distribution for the system. It does nevertheless give the most important features of the equilibrium distribution and once again joint probabilities can be built up inductively (Exercise 5.2.2).

The flow model can be extended to allow a site to contain more than one individual. Specifically, suppose that site $j (j=2,3,\ldots,J)$ may contain up to N_j individuals and let $n_j(t)$, $0 \le n_j \le N_j$, be the number of individuals at site j at time t. Further suppose that an individual moves from site j to site k with probability intensity

$$\lambda_{jk}\frac{n_j}{N_j}\frac{N_k-n_k}{N_k} \tag{5.14}$$

an individual leaves the system from site j with probability intensity

$$\mu_j \frac{n_j}{N_j} \tag{5.15}$$

and an individual arrives at site k from outside the system with probability intensity

$$\nu_k \frac{N_k - n_k}{N_k} \tag{5.16}$$

This extended model can be obtained as a limiting case of the original flow model; we simply replace site j by N_j fictitious subsites (each of which can contain at most one individual) with infinite intensities of movement between the subsites replacing site j. In this way it is easy to see that if

$$p_j(t) = \frac{E(n_j(t))}{N_j}$$

then $p_j(t)$ is the potential at time t in an electrical network constructed as follows: for $j, k = 2, 3, \ldots, J$ join nodes j and k by a wire of resistance λ_{jk}^{-1} whenever $\lambda_{jk} > 0$, join node j to nodes held at potentials 0 and 1 through wires of resistance μ_j^{-1} and ν_j^{-1} respectively, and connect node j to earth through a capacitor with capacitance N_j. The initial conditions for the electrical network will, as usual, be obtained from the initial conditions for the flow model. It is interesting to note that in the button model corresponding to this extended flow model a button performs a reversible, but not necessarily symmetric, random walk.

Unlike the network models considered in earlier chapters the flow model has few applications. The difficulty is that the symmetry condition (5.13) is too restrictive. It implies that the individuals have no innate tendency to move in any given direction and that flow of individuals through the system is the result of them being forced in at some sites and removed from other sites. This is unlikely to be the case in any of the applications discussed in Chapter 4; in a communication network, for example, we would expect a message to have a preferred direction of travel. We can of course regard the flow model as a naive description of the mechanism governing the movement of electrons in a conductor, and it then provides a physical explanation of the mathematical relationships between random walks and electrical networks obtained in Section 1.4 and the previous section.

Exercises 5.2

1. Suppose we have the solution for $p_j(t)$, $j = 2, 3, \ldots, J$, and that we are interested in finding the joint probabilities that given pairs of sites are occupied at time t. Show that these probabilities correspond to potentials

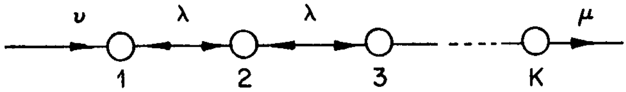

Fig. 5.1 A one-dimensional flow model

in an electrical network with $\frac{1}{2}J(J-1)$ nodes, $J-1$ of which are held at the time-varying potentials $p_j(t)$, $j=2,3,\ldots,J$.

2. In the one-dimensional flow model illustrated in Fig. 5.1 jumps take place between adjacent sites at rate λ, particles arrive at site 1 at rate ν, and leave from site K at rate μ. Deduce from the electrical analogue that in equilibrium the mean rate of flow of particles through the system is

$$[(K-1)\lambda^{-1}+\nu^{-1}+\mu^{-1}]^{-1}$$

If $\nu=\mu=\lambda$ show that sites j and $k(j<k)$ are both occupied with probability

$$\frac{(K-j)(K+1-k)}{K(K+1)}$$

and both empty with probability

$$\frac{j(k-1)}{K(K+1)}$$

3. Let $A(t)$ be the set of occupied sites in a flow model. In general the reversed process $A(-t)$ is a complicated Markov process quite unlike the original process $A(t)$. Show that in the reversed process obtained from the one-dimensional flow model illustrated in Fig. 5.1 the probability intensity that a particle leaves the system from site 1 depends not only on whether or not a particle is present at site 1 but also upon which of sites $2,3,\ldots,K$ are occupied.

5.3 INVASION MODELS

Consider the following invasion model. There are $J-1$ sites (labelled $2,3,\ldots,J$) and each site is coloured either black or white. If sites j and k are different colours then with probability intensity λ_{jk} site k invades site j—when this happens site j takes on the color of site k while site k remains the same colour. If site j is white (respectively black) then with probability intensity ν_j (respectively μ_j) it is invaded from outside the system and becomes black (respectively white). Viewed as a representation of competition between two opposing species or armies the important characteristic of the model is that the chance site j is overrun by site k depends only upon λ_{jk} and not upon the colours involved. At least initially we shall not require that the λ_{jk} be symmetric. We shall suppose that at time $t=0$ all sites in the system are white and we shall be interested in finding $p_j(t)$, the probability that at time t site j is black.

We can replace invasions from outside the system by adding two sites (labelled 0 and 1) which at time $t=0$ are white and black respectively, with

$$\begin{aligned} \lambda_{j0} &= \mu_j \qquad & \lambda_{0j} &= 0 \\ \lambda_{j1} &= \nu_j \qquad & \lambda_{1j} &= 0 \end{aligned} \qquad j = 2, 3, \ldots, J$$

and

$$\lambda_{01} = \lambda_{10} = 0$$

The formulation of the model allowed site k to invade site j only when sites j and k were different colours. Now amend the model to allow site k to invade site j at rate λ_{jk} even when sites j and k are the same colour—these additional invasions will of course result in no change of any site colour. Thus as far as the colouring of the sites of the system is concerned the amendment will not affect the stochastic behaviour of the system. It will, however, mean that invasions of site j from site k form a Poisson process of rate λ_{jk} and that as j and k vary they index independent Poisson processes.

Suppose now that we know the exact moments within the interval $(0, t)$ at which site j is invaded from site k for $j, k = 0, 1, \ldots, J$. From this information can we discover the colour of site j at time t? Consider the following method. Starting from time t look backwards through time to discover when site j was last invaded and from whence. If it was last invaded after time $t=0$ from another site of the set $\{2, 3, \ldots, J\}$, then look further back in time to discover when and from whence this site was last invaded, and so on. Remembering that the moments of invasion form realizations from independent Poisson processes it becomes clear that as we trace the origin of site j's colour backwards through time we will be following a random walk with transition intensities λ_{jk}, for $j, k = 0, 1, \ldots, J$. If this random walk reaches site 1 within a time t then site j at time t is black; otherwise it is white. Thus $p_j(t)$ is simply the probability that a random walk starting at site j and with transition rates λ_{jk}, for $j, k = 0, 1, \ldots, J$, reaches site 1 before a time t has elapsed. If the random walk on the set $\{2, 3, \ldots, J\}$ defined by the transition intensities λ_{jk}, for $j, k = 2, 3, \ldots, J$, is reversible then we can as outlined in Section 5.1 obtain an electrical analogy. The random walk will be reversible if the λ_{jk} are symmetric. It will also be reversible if the graph defined on the set $\{2, 3, \ldots, J\}$ by the λ_{jk} (with an edge joining nodes j and k if either λ_{jk} or λ_{kj} is positive) is a tree (Lemma 1.5). Even when an electrical analogy exists it is not as useful as in the last section—there is nothing in the model which can be readily related to a flow of current.

It is worth emphasizing that the analysis of the invasion model does not rely upon the random walk defined by the transition intensities λ_{jk} being reversible. The origin of a site's colour can be traced backwards through time whatever the transition intensities. There is thus a contrast with the flow model of the previous section where the analysis breaks down unless the λ_{jk} are symmetric.

Exercises 5.3

1. In the analysis of the basic invasion model it was assumed that invasions of site j from site k form a Poisson process and that for distinct pairs (j, k) the Poisson processes are independent. Show that the analysis does not depend upon the independence assumption by considering the following cases:
   (i) Whenever site j invades, it simultaneously invades every site it is adjacent to.
   (ii) When site j invades site i, site l invades site k.
   In case (i) show that an electrical analogue exists in which all the resistors joining sites have the same resistance.
2. Generalize the invasion model to allow more than two colours.
3. Consider the following stochastic model of group decision making. A group consists of n individuals, and initially individual j holds opinion (or view) v_j, $j=1,2,\ldots,n$. If at time t individuals j and k hold differing opinions then the probability that individual j is convinced by individual k and changes his opinion to that of individual k in the interval $(t, t+\delta t)$ is $\lambda_{jk}\,\delta t+o(\delta t)$. Assume that between any two individuals there exists the possibility of communication, either directly or indirectly via a chain of other individuals. Show that the group will ultimately agree on view v_j with probability α_j where α_j is the solution to the equations
$$\alpha_j \sum_k \lambda_{jk} = \sum_k \alpha_k \lambda_{kj}$$
$$\sum_k \alpha_k = 1$$
4. In the preceding exercise suppose the probability that individual j concedes to individual k in the interval $(t, t+\delta t)$ is $\lambda_{jk} f(t)\,\delta t+o(\delta t)$; for example if $f(t)$ is a decreasing function then individuals become more stubborn as time progresses. Show that the conclusion remains the same provided $\int_0^\infty f(t)\,dt$ is infinite.
5. In Exercise 5.3.3 suppose the probability that individual j concedes to individual k in the interval $(t, t+\delta t)$ is $\lambda_{jk} f_{jk}(t)\,\delta t+o(\delta t)$ where
$$f_{jk}(t)=f_{kj}(t) \qquad \text{for} \quad t>0, \quad j, k=1,2,\ldots,n$$
These rates might arise if the degree of contact between individuals varies with time. Show that the conclusion remains the same provided $f_{jk}(t)$ is a bounded function of t, $\int_0^\infty f_{jk}(t)\,dt$ is infinite, and $\alpha_j\lambda_{jk}=\alpha_k\lambda_{kj}$ for $j, k=1,2,\ldots,n$. Deduce the corresponding result when $(f_{jk}(t);\, j, k=1,2,\ldots,n)$ is itself a stochastic process.