CHAPTER 1

Markov Processes and Reversibility

In this chapter the concept of reversibility is introduced and explored, and some simple stochastic models are described. The rest of the book will be devoted to generalizations of these simple models.

The first section reviews some aspects of the theory of Markov processes which will be required in the sequel.

1.1 PRELIMINARIES ON MARKOV PROCESSES

Let $X(t)$ be a stochastic process taking values in a countable state space $\mathcal{S}$ for $t \in \mathcal{T}$. Thus $(X(t_1), X(t_2), \ldots, X(t_n))$ has a known distribution for $t_1, t_2, \ldots, t_n \in \mathcal{T}$. For a discrete time stochastic process $\mathcal{T}$ will be the integers $\mathbb{Z}$ while for a continuous time stochastic process $\mathcal{T}$ will be the real line $\mathbb{R}$. These are the only possibilities we shall consider.

If $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathcal{T}$ then the stochastic process $X(t)$ is *stationary*.

The stochastic process $X(t)$ is a *Markov* process if for $t_1 < t_2 < \cdots < t_n < t_{n+1}$ the joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n), X(t_{n+1}))$ is such that

$$P(X(t_{n+1}) = j_{n+1} \mid X(t_1) = j_1, X(t_2) = j_2, \ldots, X(t_n) = j_n)$$
$$= P(X(t_{n+1}) = j_{n+1} \mid X(t_n) = j_n)$$

whenever the conditioning event $(X(t_1) = j_1, X(t_2) = j_2, \ldots, X(t_n) = j_n)$ has positive probability. Where no confusion can arise we shall use an abbreviated notation in which the above equation becomes

$$P(j_{n+1} \mid j_1, j_2, \ldots, j_n) = P(j_{n+1} \mid j_n)$$

Thus for a Markov process the state of the process at a given time contains all the information about the past evolution of the process which is of use in predicting its future behaviour. This is the usual definition of a Markov process. An alternative equivalent definition is the following. The stochastic process $X(t)$ is a Markov process if for $t_1 < t_2 < \cdots < t_p < \cdots < t_m$, conditional on $X(t_p) = j_p$ (the present), $(X(t_1), X(t_2), \ldots, X(t_{p-1}))$ (the past) and $(X(t_{p+1}), X(t_{p+2}), \ldots, X(t_m))$ (the future) are independent (Exercise 1.1.2).

A Markov process is *time homogeneous* if $P(X(t+\tau) = k \mid X(t) = j)$ does not depend upon t, and is *irreducible* if every state in $\mathcal{S}$ can be reached from every other state. For a time homogeneous discrete time Markov process

$$p(j, k) = P(X(t+1) = k \mid X(t) = j)$$

is called the *transition probability* from state j to state k. Note that

$$\sum_{k \in \mathcal{S}} p(j, k) = 1 \qquad j \in \mathcal{S}$$

A discrete time Markov process is *periodic* if there exists an integer $\delta > 1$ such that $P(X(t+\tau) = j \mid X(t) = j) = 0$ unless τ is divisible by δ; otherwise the process is *aperiodic*. Throughout this work we shall assume that any discrete time Markov process with which we deal is time homogeneous and irreducible; we shall often additionally assume it is aperiodic. Consider then a process satisfying all these assumptions. Such a process may possess an *equilibrium distribution*, that is a collection of positive numbers $\pi(j)$, $j \in \mathcal{S}$, summing to unity that satisfy the equilibrium equations

$$\pi(j) = \sum_{k \in \mathcal{S}} \pi(k) p(k, j) \qquad j \in \mathcal{S} \tag{1.1}$$

If we can find a collection of positive numbers satisfying equations (1.1) whose sum is finite, then the collection can be normalized to produce an equilibrium distribution. When an equilibrium distribution exists it is unique and

$$\lim_{t \to \infty} P(X(t) = k \mid X(0) = j) = \pi(k) \tag{1.2}$$

so that π is the limiting distribution. Also, the proportion of time the process spends in state k during the period $[0, t]$ converges to $\pi(k)$ as $t \to \infty$, that is the process is ergodic. Further, if $P(X(0) = j) = \pi(j)$, $j \in \mathcal{S}$, then $P(X(t) = j) = \pi(j)$, $j \in \mathcal{S}$, for all $t \in \mathbb{Z}$, so that π is the stationary distribution. If an equilibrium distribution does not exist then

$$\lim_{t \to \infty} P(X(t) = k) = 0 \qquad k \in \mathcal{S}$$

and the process cannot be stationary. An equilibrium distribution will not exist if we can find a collection of positive numbers satisfying equations (1.1) whose sum is infinite. An equilibrium distribution will always exist when $\mathcal{S}$ is finite. All of the above remains true for periodic processes, except for the relation (1.2).

It is possible to construct continuous time Markov processes which exhibit extremely strange behaviour. These will be excluded; throughout this work we shall assume that any continuous time Markov process with which we deal is not only time homogeneous and irreducible but also remains in each state for a positive length of time and is incapable of passing through an infinite number of states in a finite time. Define the *transition rate* from state j to state k to be

$$q(j,k)=\lim_{\tau\to 0}\frac{P(X(t+\tau)=k \mid X(t)=j)}{\tau} \qquad j\neq k$$

It will be convenient to let $q(j,j)=0$. For continuous time processes the equilibrium equations are

$$\pi(j)\sum_{k\in\mathcal{S}} q(j,k)=\sum_{k\in\mathcal{S}}\pi(k)q(k,j) \qquad j\in\mathcal{S} \tag{1.3}$$

and an equilibrium distribution is a collection of positive numbers $\pi(j)$, $j\in\mathcal{S}$, summing to unity which satisfy the equilibrium equations. As for discrete time processes an equilibrium distribution is unique if it exists and is then both the limiting and the stationary distribution. Further, the process is ergodic. If one does not exist then

$$\lim_{t\to\infty} P(X(t)=k)=0 \qquad k\in\mathcal{S}$$

An equilibrium distribution will not exist if we can find a collection of positive numbers satisfying equations (1.3) whose sum is infinite. When $\mathcal{S}$ is finite an equilibrium distribution will always exist.

A discrete time Markov process is sometimes called a Markov chain. We shall use this terminology so that from now on when we refer to a Markov process it will be a continuous time process. We shall often refer to a stationary Markov chain or process as being in equilibrium.

A Markov process remains in state j for a length of time which is exponentially distributed with parameter

$$q(j)=\sum_{k\in\mathcal{S}} q(j,k)$$

When it leaves state j it moves to state k with probability

$$p(j,k)=\frac{q(j,k)}{q(j)} \tag{1.4}$$

There is thus a natural way to associate a Markov chain $X^J(t)$ with a Markov process $X(t)$. Let $X^J(0)$ be $X(0)$, let $X^J(1)$ be the next state the Markov process $X(t)$ moves to after time $t=0$, let $X^J(2)$ be the next state after that, and so on. The Markov chain $X^J(t)$ is called the *jump chain* of the Markov process $X(t)$, and its transition probabilities are given by the relation (1.4). The equilibrium distribution of a jump chain will in general be different from that of the Markov process generating it (Exercise 1.1.5), essentially because the jump chain ignores the length of time the process remains in each state. The initial distribution at time $t=0$ of the jump chain of a stationary Markov process will be the equilibrium distribution of the Markov process, and thus the jump chain will not in general be stationary.

Exercises 1.1

1. Let $Z(t)$, $t \in \mathbb{Z}$, be a sequence of independent identically distributed random variables with $P(Z(t)=0)=P(Z(t)=1)=\frac{1}{2}$. Define the stochastic process $X(t)$ with $\mathcal{S}=\{0,1,2,\ldots,6\}$ and $\mathcal{T}=\mathbb{Z}$ by $X(t)=Z(t-1)+2Z(t)+3Z(t+1)$.
   (a) Determine $P(X(0)=1,\ X(1)=3,\ X(2)=2)$ and $P(X(1)=3,\ X(2)=2)$.
   (b) Determine $P(X(2)=2 \mid X(0)=1,\ X(1)=3)$ and $P(X(2)=2 \mid X(1)=3)$. Deduce that the process $X(t)$ is not Markov.
2. Establish the equivalence of the following statements:
   (i) For all $t_1<t_2<\cdots<t_n<t_{n+1}$,
   $$P(j_{n+1} \mid j_1, j_2, \ldots, j_n)=P(j_{n+1} \mid j_n)$$
   (ii) For all $t_1<t_2<\cdots<t_p<\cdots<t_m$,
   $$P(j_1, j_2, \ldots, j_{p-1}, j_{p+1}, j_{p+2}, \ldots, j_m \mid j_p) = P(j_1, j_2, \ldots, j_{p-1} \mid j_p)P(j_{p+1}, j_{p+2}, \ldots, j_m \mid j_p)$$
3. If a Markov process has an equilibrium distribution show that the convergence to it expressed in the relation (1.2) is uniform over states $k \in \mathcal{S}$.
4. Consider the Markov process with state space $\mathcal{S}=\{0,1,2,\ldots\}$ and with transition rates
   $$q(j,k)=\begin{cases} a^j & k=j+1 \\ b & k=0 \\ 0 & \text{otherwise} \end{cases}$$
   If $a>1$ this process is capable of passing through an infinite number of states in finite time. Find the equilibrium distribution when $a \le 1$ and $b>0$. Observe that one does not exist when $0<a\le 1$ and $b=0$.
5. It is possible for a Markov process to possess an equilibrium distribution and for its jump chain not to, and vice versa. Show that if a Markov process has equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, then its jump chain has an equilibrium distribution if and only if
   $$B^{-1}=\sum_{j\in\mathcal{S}} \pi(j)q(j)$$

is finite, in which case the equilibrium distribution of the jump chain is

$$\pi^J(j) = B\pi(j)q(j).$$

Observe that if $q(j)$ does not depend upon j, so that the points in time at which jumps take place form a Poisson process, then the jump chain and the process have the same equilibrium distribution.

1.2 REVERSIBILITY

Some stochastic processes have the property that when the direction of time is reversed the behaviour of the process remains the same. Speaking intuitively, if we take a film of such a process and then run the film backwards the resulting process will be statistically indistinguishable from the original process. This property is described formally in the following definition.

Definition

A stochastic process $X(t)$ is *reversible* if $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathcal{T}$.

In the next section we shall give examples of reversible processes and in later sections we shall discuss some of the less obvious consequences of the above definition; but first let us derive some of the basic properties of reversible processes.

Lemma 1.1. *A reversible process is stationary.*

Proof. Since $X(t)$ is reversible both $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau))$ have the same distribution as $(X(-t_1), X(-t_2), \ldots, X(-t_n))$. Hence $X(t)$ is stationary.

For a stationary Markov chain or process there exist simple necessary and sufficient conditions for reversibility given in terms of the equilibrium distribution and the transition probabilities or rates. These conditions are obtained in Theorems 1.2 and 1.3 and are called the detailed balance conditions; they should be contrasted with the equilibrium equations, which are sometimes called the full balance conditions.

Theorem 1.2. *A stationary Markov chain is reversible if and only if there exists a collection of positive numbers $\pi(j)$, $j \in \mathcal{S}$, summing to unity that satisfy the detailed balance conditions*

$$\pi(j)p(j, k) = \pi(k)p(k, j) \qquad j, k \in \mathcal{S} \tag{1.5}$$

When there exists such a collection $\pi(j)$, $j \in \mathcal{S}$, it is the equilibrium distribution of the process.

Proof. First suppose that the process is reversible. Since the process is stationary $P(X(t)=j)$ does not depend upon t. Let $\pi(j)=P(X(t)=j)$; thus $\pi(j)$, $j \in \mathcal{S}$, is a collection of positive numbers summing to unity. Since the process is reversible

$$P(X(t)=j, X(t+1)=k)=P(X(t)=k, X(t+1)=j)$$

and so

$$\pi(j)p(j,k)=\pi(k)p(k,j)$$

Conversely, suppose there exists a collection of positive numbers $\pi(j)$, $j \in \mathcal{S}$, summing to unity satisfying the detailed balance conditions. Summing equations (1.5) over k we obtain

$$\pi(j)\sum_{k \in \mathcal{S}} p(j,k)=\sum_{k \in \mathcal{S}} \pi(k)p(k,j) \qquad j \in \mathcal{S}$$

which reduce to the equilibrium equations (1.1). Hence the collection $\pi(j)$, $j \in \mathcal{S}$, is the equilibrium distribution of the process. Consider now a sequence of states $j_0, j_1, \ldots, j_m$. Then

$$P(X(t)=j_0, X(t+1)=j_1, \ldots, X(t+m)=j_m)$$
$$=\pi(j_0)p(j_0,j_1)p(j_1,j_2)\cdots p(j_{m-1},j_m)$$

and

$$P(X(t')=j_m, X(t'+1)=j_{m-1}, \ldots, X(t'+m)=j_0)$$
$$=\pi(j_m)p(j_m,j_{m-1})p(j_{m-1},j_{m-2})\cdots p(j_1,j_0)$$

But the detailed balance conditions (1.5) imply that the right-hand sides of the last two identities are equal. Hence, letting $\tau=t+t'+m$, $(X(t), X(t+1), \ldots, X(t+m))$ has the same distribution as $(X(\tau-t), X(\tau-t-1), \ldots, X(\tau-t-m))$, and from this we can deduce that $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(\tau-t_1), X(\tau-t_2), \ldots, X(\tau-t_n))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{Z}$.

The detailed balance conditions (1.5) imply that if $p(j,k)$ is positive then so is $p(k,j)$. Less obvious, but interesting, consequences for the matrix of transition probabilities are contained in Exercises 1.2.4 and 1.2.5.

Theorem 1.2 has a direct analogue for continuous time processes.

Theorem 1.3. *A stationary Markov process is reversible if and only if there exists a collection of positive numbers $\pi(j)$, $j \in \mathcal{S}$, summing to unity that satisfy* *the detailed balance conditions*

$$\pi(j)q(j, k) = \pi(k)q(k, j) \qquad j, k \in \mathcal{S} \tag{1.6}$$

When there exists such a collection $\pi(j)$, $j \in \mathcal{S}$, it is the equilibrium distribution of the process.

Proof. First suppose the process is reversible, and let $\pi(j) = P(X(t) = j)$. Then

$$P(X(t) = j, X(t+\tau) = k) = P(X(t) = k, X(t+\tau) = j)$$

and so

$$\pi(j)\frac{P(X(t+\tau) = k \mid X(t) = j)}{\tau} = \pi(k)\frac{P(X(t+\tau) = j \mid X(t) = k)}{\tau}$$

Letting $\tau \to 0$ we obtain the relation (1.6).

Conversely, suppose there exists the collection $\pi(j)$, $j \in \mathcal{S}$, satisfying the detailed balance conditions. Summing equations (1.6) over k gives the equilibrium equations (1.3), and hence the collection $\pi(j)$, $j \in \mathcal{S}$, is the equilibrium distribution. Consider now the behaviour of the process $X(t)$ for $t \in [-T, T]$. The process may start at time $t = -T$ in state j_1 and remain in this state for a period h_1 before jumping to state j_2. Suppose it now remains in state j_2 for a period h_2 before jumping to state j_3, and so on, until it arrives in state j_m where it remains until time $t = T$, a period of h_m, say. Now the probability density of the random variable h_1 is

$$q(j_1)e^{-q(j_1)h_1}$$

and the probability that j_2 is the next state after j_1 is

$$\frac{q(j_1, j_2)}{q(j_1)}$$

Similarly, we can calculate the density of h_2 and the probability that j_3 is the next state after j_2, and so on. The probability that the process remains in state j_m for a period of at least h_m is

$$e^{-q(j_m)h_m}$$

Thus the probability density of the behaviour described is

$$\pi(j_1)e^{-q(j_1)h_1}q(j_1, j_2)e^{-q(j_2)h_2}q(j_2, j_3)e^{-q(j_3)h_3} \cdots q(j_{m-1}, j_m)e^{-q(j_m)h_m} \tag{1.7}$$

This is a density with respect to $h_1, h_2, \ldots, h_m$. To obtain a probability it must be integrated over a region of values for $h_1, h_2, \ldots, h_m$ satisfying the constraint $h_1 + h_2 + \cdots + h_m = 2T$. Now the relation (1.6) implies that

$$\pi(j_1)q(j_1, j_2)q(j_2, j_3) \cdots q(j_{m-1}, j_m) = \pi(j_m)q(j_m, j_{m-1}) \cdots q(j_3, j_2)q(j_2, j_1)$$

and hence that expression (1.7) is equal to the probability density that the process starts at time $t=-T$ in state j_m, that it remains in this state for a period h_m before jumping to state j_{m-1}, and so on, until it arrives in state j_1 which it remains in until time $t=T$, a period of h_1. Thus the probabilistic behaviour of $X(t)$ is precisely the same as that of $X(-t)$ on the interval $[-T, T]$. Thus $(X(t_1), X(t_2), \ldots, X(t_m))$ has the same distribution as $(X(-t_1), X(-t_2), \ldots, X(-t_m))$, but this has the same distribution as $(X(\tau-t_1), X(\tau-t_2), \ldots, X(\tau-t_m))$ because $X(t)$ is stationary; and so the theorem is proved.

A collection of positive numbers satisfying the detailed balance conditions whose sum is finite can of course be normalized to produce an equilibrium distribution. Lemma 1.1 shows that a Markov process which is not stationary is not reversible, even if the detailed balance condition can be satisfied.

The term $\pi(j)q(j, k)$ is called the *probability flux* from state j to state k; in equilibrium the probability that in the interval $(t, t+\delta t)$ the process jumps from state j to state k is $\pi(j)q(j, k)\,\delta t+o(\delta t)$. The detailed balance condition (1.6) requires that the probability flux from state j to k should equal that from state k to j. The full balance condition (1.3) requires that the probability flux out of state j should equal that into state j. These relationships can perhaps be more easily visualized if we associate a graph G with the Markov process as follows: let the set of vertices of the graph be $\mathcal{S}$, the set of states, and let there be an edge joining vertices j and k if either $q(j, k)$ or $q(k, j)$ is positive. Thus the Markov process can be regarded as a random walk on the graph G. Note that the assumed irreducibility of the process implies that the graph is connected. Define a cut to be a division of $\mathcal{S}$ into complementary sets $\mathcal{A}$ and $\mathcal{S}-\mathcal{A}$.

Lemma 1.4. *For a stationary Markov process the probability flux each way across a cut balances. That is for any* $\mathcal{A}\subset\mathcal{S}$,

$$\sum_{j\in\mathcal{A}}\sum_{k\in\mathcal{S}-\mathcal{A}}\pi(j)q(j, k)=\sum_{j\in\mathcal{A}}\sum_{k\in\mathcal{S}-\mathcal{A}}\pi(k)q(k, j) \tag{1.8}$$

Proof. Summing the full balance condition (1.3) over $j\in\mathcal{A}$ gives

$$\sum_{j\in\mathcal{A}}\sum_{k\in\mathcal{S}}\pi(j)q(j, k)=\sum_{j\in\mathcal{A}}\sum_{k\in\mathcal{S}}\pi(k)q(k, j)$$

The result follows by subtracting the identity

$$\sum_{j\in\mathcal{A}}\sum_{k\in\mathcal{A}}\pi(j)q(j, k)=\sum_{j\in\mathcal{A}}\sum_{k\in\mathcal{A}}\pi(k)q(k, j)$$

Note that if $\mathcal{A}=\{j\}$ then equations (1.8) reduce to the equilibrium equations (1.3).

Lemma 1.5. *If the graph G associated with a stationary Markov process is a tree, then the process is reversible.*

Proof. If j and k are not linked by an edge of the graph G the detailed balance condition (1.6) is satisfied trivially. If j and k are linked by an edge then removal of this edge cuts the graph G into two unconnected components, since G is a tree. Thus Lemma 1.4 shows that the detailed balance condition is satisfied.

Lemma 1.5 gives a sufficient condition for a process to be reversible but, as we shall see later, it is by no means necessary.

It can be shown that the number of transitions from state j to k per unit time calculated over the period $(0, t)$ converges to $\pi(j)q(j, k)$ as $t \to \infty$. This fact provides an alternative proof of Lemmas 1.4 and 1.5 since the number of transitions each way across a cut in the period $(0, t)$ cannot differ by more than one.

Lemmas 1.4 and 1.5 have obvious counterparts for Markov chains.

Exercises 1.2

1. Consider the stationary Markov process $X(t)$ with $\mathcal{S} = \{1, 2\}$, $q(1, 2) = 1$, $q(2, 1) = \frac{1}{2}$. Show that $X(t)$ is reversible. Observe that a film of the process, run either forwards or backwards, will show the process alternating between states with the periods in states 1 and 2 having means 1 and 2 respectively. There is a minor difficulty here which should be pointed out. Suppose the process jumps from state 1 to 2 at time t_0. Is $X(t_0) = 1$ or 2? The usual convention is that if $X(t)$ jumps at time t_0 then $X(t_0)$ is taken to be the new state, so that the process is right continuous. The difficulty is that if the film run forwards is right continuous then the film run backwards will be left continuous. The difficulty is avoided if we adopt the convention that $X(t_0)$ is equally likely to be the old or the new state. Such fine differences would of course be hard to detect (the finite dimensional distributions do not manage it), and will not concern us from now on. When, later, we speak of the instant in time just preceding (respectively, following) a transition we shall be implicitly appealing to the left (respectively, right) continuous version of the process.
2. Show that the stochastic process $X(t)$ defined in Exercise 1.1.1 is not reversible.
3. Suppose that the points $s_i \in \mathbb{R}$, $i = \ldots, -1, 0, 1, 2, \ldots$, form a Poisson process and define

$$X(t) = \sum_{i=-\infty}^{+\infty} a(s_i - t)$$

Show that $X(t)$ is reversible if

$$a(s)=\begin{cases}1 & -1<s\leq 0\\ 0 & \text{otherwise}\end{cases}$$

and is not reversible if

$$a(s)=\begin{cases}2 & -1<s\leq 0\\ 1 & -2<s\leq -1\\ 0 & \text{otherwise}\end{cases}$$

4. Show that a stationary Markov chain is reversible if and only if the matrix of transition probabilities can be written as the product of a symmetric and a diagonal matrix.
5. Show that the matrix of transition probabilities of a reversible Markov chain can be written in the form $D^{-1}AD$ where D is diagonal and A symmetric. Deduce that it has real eigenvalues (the converse is false as the next exercise shows).
6. Consider a stationary Markov chain with the following matrix of transition probabilities:

$$\begin{pmatrix}0 & 1 & 0\\ \frac{3}{4} & 0 & \frac{1}{4}\\ 1 & 0 & 0\end{pmatrix}$$

   Show that the process is not reversible even though the matrix has real eigenvalues.
7. Suppose a Markov process and its jump chain both possess equilibrium distributions. Observe that the equilibrium probability that the jump chain is in state j, found in Exercise 1.1.5, is proportional to the probability flux out of, or, equivalently, the probability flux into, state j in the Markov process. Show that the transition rates of the Markov process satisfy the detailed balance conditions if and only if the transition probabilities of the jump chain do.
8. If $X_1(t)$ and $X_2(t)$ are independent reversible Markov processes show that $(X_1(t), X_2(t))$ is a reversible Markov process.
9. If $X(t)$ is a reversible stochastic process show that so is $Y(t)=f[X(t)]$ for any function f.

1.3 BIRTH AND DEATH PROCESSES

The simplest examples of reversible processes are provided by Markov processes for which the state space $\mathscr{S}$ is $\{0, 1, 2, \ldots, K\}$, with K possibly infinite, and $q(j, k)=0$ unless $|j-k|=1$. These are called birth and death processes, since the only possible transitions from state j are to $j-1$ (a death) or $j+1$ (a birth). A stationary birth and death process is reversible, by Lemma 1.5. The detailed balance condition states that the equilibrium distribution of a stationary birth and death process satisfies

$$\pi(j)q(j, j-1) = \pi(j-1)q(j-1, j)$$

and hence is given by

$$\pi(j) = \pi(0) \prod_{r=1}^{j} \frac{q(r-1, r)}{q(r, r-1)} \tag{1.9}$$

where $\pi(0)$ must be chosen so that $\pi(j)$, $j = 0, 1, 2, \ldots$, sum to unity. If $\pi(0)$ cannot be so chosen then the process does not possess an equilibrium distribution and cannot be stationary. We will now discuss some simple examples of birth and death processes.

The simple queue. Suppose that the stream of customers arriving at a queue (the arrival process) forms a Poisson process of rate ν. Suppose further that there is a single server and that customers' service times are independent of each other and of the arrival process and are exponentially distributed with mean μ^{-1}. Such a queue is called simple or $M/M/1$, the M's indicating the memoryless (exponential) character of the interarrival and service times and the final digit indicating the number of servers. Let $n(t)$ be the number of customers in the queue at time t, including the customer being served. Then it follows from our description of the queue that $n(t)$ is a birth and death process with transition rates

$$q(j, j-1) = \mu \qquad j = 1, 2, \ldots$$
$$q(j, j+1) = \nu \qquad j = 0, 1, \ldots$$

If the arrival rate ν is less than the service rate μ the process has an equilibrium distribution which is, from equation (1.9),

$$\pi(j) = \left(1 - \frac{\nu}{\mu}\right)\left(\frac{\nu}{\mu}\right)^j \tag{1.10}$$

Thus in equilibrium the number in the queue has a geometric distribution with mean $\nu/(\mu - \nu)$.

This result can be used to obtain another distribution of interest, the distribution of the waiting time of a customer. We shall define waiting time to include service time, so that it is the period between a customer's arrival at and departure from the queue. Consider now a typical customer arriving at the queue and let W be his waiting time. Assume for the moment that the probability he finds j customers already present in the queue is $\pi(j)$. With the queue discipline first come first served,

$$P(W \le w) = \sum_{j=0}^{\infty} \pi(j) P\left(\sum_{r=1}^{j+1} S_r \le w\right) \tag{1.11}$$

where $S_1, S_2, \ldots$ are independent exponentially distributed random variables with mean μ^{-1}. After some reduction (Exercise 1.3.1) this shows that W is exponentially distributed with mean $(\mu-\nu)^{-1}$.

Is it valid to assume that when a typical customer arrives at the queue he finds it in equilibrium? This assumption can be made when the arrival process is Poisson, although we must be careful about the interpretation of a typical customer. If we observe a customer arriving at time t_0 and we know nothing other than this about arrival times or about the state of the queue, then we shall call this customer typical. When the arrival process is Poisson the interval between t_0 and the preceding arrival has an exponential distribution, and indeed the arrival process up until time t_0 has the same probabilistic description as it would have if t_0 were just a fixed instant in time. Hence the customer arriving at time t_0 finds the queue in equilibrium. (The concept of a typical customer is investigated further in Exercise 1.3.7.)

There is an alternative approach to this result which is of greater generality and will be of use later. The probability that in the interval $(t_0, t_0+\delta t)$ a single customer arrives and finds j customers already present in the queue is

$$\pi(j)q(j, j+1)\,\delta t+o(\delta t)$$

The probability that in the interval $(t_0, t_0+\delta t)$ a single customer arrives is

$$\sum_{j=0}^{\infty} \pi(j)q(j, j+1)\,\delta t+o(\delta t)$$

Thus given that a single customer arrives in the interval $(t_0, t_0+\delta t)$ the conditional probability that he finds j customers already present in the queue is

$$\frac{\pi(j)q(j, j+1)\,\delta t+o(\delta t)}{\sum_{j=0}^{\infty} \pi(j)q(j, j+1)\,\delta t+o(\delta t)}$$

As $\delta t \to 0$ this conditional probability tends to the ratio

$$\frac{\pi(j)q(j, j+1)}{\sum_{j=0}^{\infty} \pi(j)q(j, j+1)}$$

The numerator is the probability flux that a customer arrives to find j customers already present in the queue, and the denominator is the probability flux that a customer arrives. Thus this ratio is also the limit as $t \to \infty$ of the proportion of arrivals in the period $(0, t)$ who find j customers already present in the queue. Since $q(j, j+1)=\nu$ the ratio is simply $\pi(j)$.

The above approach is of use whenever a stochastic process is observed at just those points in time marked by some special event. Exercises 1.1.5, 1.3.6, and 1.3.9 provide further examples.

For a simple queue the mean number in the queue $E(n)$, the mean waiting time of a customer $E(W)$, and the mean time between successive arrivals ν^{-1} are related by the identity

$$E(n)=\nu E(W). \tag{1.12}$$

This, Little's result, holds for much more general systems—the arrival process need not be Poisson, service times need not be independent, and indeed the system may bear little resemblance to a queue at all. It has the nature of an accounting identity; we can count time spent in a system either by adding it up over the customers who pass through the system or by integrating the number in the system over time. We shall not prove Little's result, although we shall occasionally use it. For our purposes it will be enough to record that equation (1.12) holds whenever there is a stationary Markov process $X(t)$ such that the number in the system at time t, $n(t)$, is a function of $X(t)$. The expectation $E(W)$ can be regarded as the mean time spent in the system by a typical customer or as the limit as $m\to\infty$ of the average time spent in the system by the first m customers to enter the system after time t. Similarly, ν can be regarded as the reciprocal of the mean interarrival period preceding the arrival of a typical customer or as the limit as $t\to\infty$ of the number of customers to arrive per unit time calculated over the period $(0, t)$. In equilibrium the probability flux that a customer arrives is ν. When the arrival process is not Poisson we shall call ν the mean arrival rate.

A telephone exchange. Suppose that calls are initiated at points in time which form a Poisson process of rate ν, but that the exchange has only K lines so that a call initiated when K calls are already in progress is lost. Further suppose that calls which are connected last for lengths of time which are independent and exponentially distributed with mean μ^{-1}. Then the number of calls in progress at time t is a birth and death process with transition rates

$$q(j, j-1)=j\mu \qquad j=1,2,\ldots,K$$

$$q(j, j+1)=\nu \qquad j=0,1,\ldots,K-1$$

The equilibrium distribution over the state space $\mathcal{S}=\{0,1,\ldots,K\}$ is

$$\pi(j)=\pi(0)\frac{1}{j!}\left(\frac{\nu}{\mu}\right)^j$$

Thus in equilibrium the number of calls in progress has a truncated Poisson distribution.

The probability that a typical call will be lost is

$$\pi(K)=\frac{(1/K!)(\nu/\mu)^K}{\sum_{j=0}^{K}(1/j!)(\nu/\mu)^j} \tag{1.13}$$

This, Erlang's formula, also gives the limiting proportion of calls lost.

The simple birth, death, and immigration process. Suppose that individuals immigrate at rate ν, give birth to additional individuals at rate λ, and die at rate μ so that

$$q(j, j-1) = j\mu \qquad j = 1, 2, \ldots$$
$$q(j, j+1) = \nu + j\lambda \qquad j = 0, 1, \ldots$$

These transition rates correspond to the assumptions that the lifetimes of individuals are independent of each other and of the immigration process and that during an individual's lifetime the points in time at which it gives birth form a Poisson process independent of other lifetimes and of the immigration process. It is often tedious to specify precisely the assumptions underlying a model; where the assumptions are clear from the context or from the structure of a Markov process we shall often fail to list them. It follows from equation (1.9) that when $\lambda < \mu$ the equilibrium distribution for the number of individuals alive is

$$\pi(j) = \left(1 - \frac{\lambda}{\mu}\right)^{\nu/\lambda} \binom{\frac{\nu}{\lambda} + j - 1}{j} \left(\frac{\lambda}{\mu}\right)^j \tag{1.14}$$

where

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r(r-1)\cdots 1}$$

This distribution is an example of the negative binomial distribution; its mean is $\nu/(\mu - \lambda)$ and its variance is $\nu\mu/(\mu - \lambda)^2$. When $\lambda = \nu$ it reduces to the geometric distribution (1.10).

Exercises 1.3

1. Relation (1.11) shows that W is the sum of $j+1$ independent exponentially distributed random variables, where j itself is a random variable with a geometric distribution. By considering the Markov process with three states and transition rates $q(1, 2) = q(2, 1) = \nu$, $q(1, 3) = q(2, 3) = \mu - \nu$, show that W is exponentially distributed with mean $(\mu - \nu)^{-1}$.
2. Suppose the simple queue described above is amended by the requirement that any customer who arrives when there are K customers already present must leave immediately without service. Show that in equilibrium the probability that this amended queue contains n customers is just the conditional probability that the simple queue contains n customers given that it contains not more than K customers.
3. Show that for an $M/M/s$ queue the number in the queue is a birth and

death process whose equilibrium distribution is determined by

$$\pi(j) = \pi(0)\left(\frac{\nu}{\mu}\right)^j \frac{1}{j!} \qquad j = 1, 2, \ldots, s$$

$$\pi(j) = \pi(s)\left(\frac{\nu}{s\mu}\right)^{j-s} \qquad j = s+1, s+2, \ldots$$

provided $\nu < s\mu$. The ratio $\rho = \nu/s\mu$ is called the traffic intensity. Show that if a typical customer arrives to find all the servers busy then, with the queue discipline first come first served, his queueing time (the period of time until his service commences) is exponentially distributed with mean $(s\mu - \nu)^{-1}$.

4. Suppose the number of customers in an $M/M/1$ queue is observed at those instants in time at which a customer is about to arrive. Show that the resulting discrete time process is a Markov chain with transition probabilities

$$p(j, k) = \frac{\nu}{\nu + \mu}\left(\frac{\mu}{\nu + \mu}\right)^{j-k+1} \qquad 0 \le k \le j+1$$

Verify that the equilibrium distribution is given by the expression (1.10).

5. The Poisson assumption in the telephone exchange model may be adequate if the source population of subscribers is very large. If the source population is of finite size $M(>K)$, it may be more reasonable to let

$$q(j, j+1) = \lambda(M - j) \qquad j = 0, 1, \ldots, K-1$$

Show that the equilibrium distribution will then be given by

$$\pi(j) = \pi(0)\binom{M}{j}\left(\frac{\lambda}{\mu}\right)^j \qquad j = 0, 1, \ldots, K$$

6. Consider the finite source telephone exchange model of the previous exercise. Suppose the number of busy lines is observed at those instants in time at which a call is about to be initiated. Write down the transition probabilities of the resulting Markov chain. By considering the probability flux $\pi(j)q(j, j+1)$ that a call is initiated while j lines are busy show that the equilibrium distribution of the Markov chain is given by

$$\pi'(j) = \pi'(0)\binom{M-1}{j}\left(\frac{\lambda}{\mu}\right)^j \qquad j = 0, 1, 2, \ldots, K$$

Comparing this distribution with the one obtained in the preceding exercise we see that the number of busy lines found by a subscriber when he attempts to make a call has the same distribution as the number of busy lines at a fixed instant in time in a system with one less subscriber.

7. Suppose that a typical customer arrives at an $M/M/1$ queue at time t_0. Show that the mth customer to arrive after time t_0 finds the queue in equilibrium, for $m = 1, 2, \ldots$. In contrast, the first customer to arrive after a fixed instant in time does not find the queue in equilibrium, since the interarrival period preceding his arrival is the sum of two exponential random variables. This customer is not typical: the way in which he has been chosen provides us with information about the time of previous arrivals. Show that the probability this customer finds j customers in the queue is

$$\begin{array}{ll} (1-\rho)(1+\rho) & j=0 \\ (1-\rho)\rho^{j+1} & j=1,2,\ldots \end{array}$$

where $\rho = \nu/\mu$.

8. A stack is a form of queue in which the server devotes his entire attention to the customer who last arrived at the queue. Thus when a customer arrives his service is started immediately, but is interrupted if another customer arrives before its completion. Suppose that customers are of I types, that the stream of customers of type i arriving at the queue forms a Poisson process of rate ν_i, and that the service times of these customers are exponentially distributed with parameter μ_i. Construct a Markov process to represent the queue and show that the graph associated with the process is a tree. Show that if

$$\rho = \sum_{i=1}^{I} \frac{\nu_i}{\mu_i} < 1$$

the equilibrium probability that there are n customers in the queue with the rth customer being of type $t(r)$ is

$$(1-\rho) \prod_{r=1}^{n} \frac{\nu_{t(r)}}{\mu_{t(r)}}$$

Deduce that in equilibrium the number of customers in the queue has the same distribution as for the simple queue with $\nu/\mu = \rho$.

9. Consider the points in time at which new individuals appear, either through immigration or birth, in the simple birth, death, and immigration process. Show that the mean time between such appearances is $(\mu - \lambda)/\nu\mu$ by using Little's result (1.12). Equivalently show that the mean appearance rate is $\nu\mu/(\mu - \lambda)$ by calculating the probability flux that a new individual appears. Show that when a new individual appears the number of individuals he finds already alive has a negative binomial distribution with mean $(\nu + \lambda)/(\mu - \lambda)$. Conditional on the new individual having been born show that the number of individuals he finds already alive, excluding his parent, has the same negative binomial distribution.

1.4 THE EHRENFEST MODEL

One particular example of a birth and death process is worth special study; it was introduced early in the century to help explain the apparent paradox between reversibility and the phenomenon of increasing entropy. The model can be described as follows. There are K particles distributed between two containers (Fig. 1.1). Particles behave independently and change container at rate λ. Thus $X(t)$, the number of particles in container 1 at time t, is a Markov process with transition rates

$$q(j, j-1) = j\lambda \qquad j = 1, 2, \ldots, K$$

$$q(j, j+1) = (K-j)\lambda \qquad j = 0, 1, \ldots, K-1$$

The equilibrium distribution can be deduced from equation (1.9) and is

$$\pi(j) = 2^{-K}\binom{K}{j}$$

The process in equilibrium is reversible and thus, assuming K is even,

$$P(X(t) = K, X(t+\tau) = \tfrac{1}{2}K) = P(X(t) = \tfrac{1}{2}K, X(t+\tau) = K) \qquad (1.15)$$

The equilibrium distribution shows that states which allocate particles fairly evenly between the two containers are much more likely than states which allocate most of the particles to one container. Hence the conditional probability $P(X(t+\tau) = \frac{1}{2}K \mid X(t) = K)$ is much greater than $P(X(t+\tau) = K \mid X(t) = \frac{1}{2}K)$. If the process starts with all the particles in one container then it is quite likely that after a period the particles will be shared evenly between the two containers. On the other hand, if the process starts with the particles shared evenly between the containers it is extremely unlikely that after a period the particles will all be in one container. The lack of symmetry exhibited by the conditional probabilities is quite compatible with reversibility. It is joint probabilities, such as those appearing in equation (1.15), which reversibility requires to be symmetric.

The asymmetry of the conditional probabilities, and more generally the phenomenon of increasing entropy, is a symptom of the approach to equilibrium of a system not initially in equilibrium. Consider a Markov process $X(t)$ with a finite state space. Let

$$u_j(t) = P(X(t) = j)$$

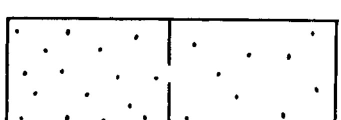

Fig. 1.1 The Ehrenfest model

and suppose that the initial distribution, $u_j(0)$, $j \in \mathcal{S}$, may not be the equilibrium distribution. Considering the possible events in the time interval $(t, t+\delta t)$ leads to the equation

$$u_j(t+\delta t) = \sum_{k \in \mathcal{S}} u_k(t) q(k, j)\, \delta t + u_j(t)\left(1 - \sum_{k \in \mathcal{S}} q(j, k)\, \delta t\right) + o(\delta t)$$

and hence the forward equations

$$\frac{\mathrm{d}}{\mathrm{d}t} u_j(t) = \sum_{k \in \mathcal{S}} (u_k(t) q(k, j) - u_j(t) q(j, k)) \quad j \in \mathcal{S} \tag{1.16}$$

The solution to these equations must satisfy the initial conditions at time $t = 0$, and tends to the equilibrium distribution as $t \to \infty$. Now let

$$H(t) = \sum_{j \in \mathcal{S}} \pi(j) h\left(\frac{u_j(t)}{\pi(j)}\right)$$

where $h(x)$ is a strictly concave function. Thus $H(t)$ is a function of the distribution over states at time t, $u_j(t)$, $j \in \mathcal{S}$. If the initial distribution is the equilibrium distribution, then $H(t)$ takes a constant value. Otherwise $H(t)$ increases monotonically to this constant value, as the next theorem shows.

Theorem 1.6. *If the initial distribution is not the equilibrium distribution, then the function $H(t)$, $t > 0$, is strictly increasing.*

Proof. For fixed $\tau > 0$ let

$$p(j, k) = P(X(t+\tau) = k \mid X(t) = j)$$

Thus

$$u_k(t+\tau) = \sum_j u_j(t) p(j, k)$$

and

$$\pi(k) = \sum_j \pi(j) p(j, k)$$

Let

$$a(k, j) = \frac{\pi(j) p(j, k)}{\pi(k)} \tag{1.17}$$

Thus $a(k, j) > 0$, $\sum_j a(k, j) = 1$. Also

$$\frac{u_k(t+\tau)}{\pi(k)} = \sum_j \frac{u_j(t) p(j, k)}{\pi(k)}$$

$$= \sum_j a(k, j) \frac{u_j(t)}{\pi(j)} \tag{1.18}$$

Now since $h(x)$ is strictly concave

$$h\left(\sum_j a(k, j)x_j\right) > \sum_j a(k, j)h(x_j) \tag{1.19}$$

unless x_j, $j \in \mathcal{S}$, are all equal. Using successively relations (1.18), (1.19), and (1.17) we have that unless $u_j(t) = \pi(j)$, $j \in \mathcal{S}$,

$$\begin{aligned} H(t+\tau) &= \sum_k \pi(k)h\left(\frac{u_k(t+\tau)}{\pi(k)}\right) \\ &= \sum_k \pi(k)h\left(\sum_j a(k, j)\frac{u_j(t)}{\pi(j)}\right) \\ &> \sum_j \sum_k \pi(k)a(k, j)h\left(\frac{u_j(t)}{\pi(j)}\right) \\ &= \sum_j \sum_k \pi(j)p(j, k)h\left(\frac{u_j(t)}{\pi(j)}\right) \\ &= H(t) \end{aligned}$$

The theorem has a counterpart for Markov chains which is established in the same way.

An important special case of the theorem arises with the concave function $h(x) = -x \log x$. Then

$$H(t) = -\sum_j u_j(t) \log \frac{u_j(t)}{\pi(j)}$$

This quantity is called the statistical entropy, or the entropy of the distribution $u_j(t)$, $j \in \mathcal{S}$, with respect to the distribution $\pi(j)$, $j \in \mathcal{S}$.

The monotonic increase of the function $H(t)$ is a consequence of the convergence of the distribution $u_j(t)$, $j \in \mathcal{S}$, to the equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$. It will occur whether or not the Markov process $X(t)$ has transition rates which satisfy the detailed balance conditions (1.6), provided only that the process is not in equilibrium. On the other hand, reversibility is essentially a property which a process in equilibrium may or may not possess, and in either case the function $H(t)$ is constant just because the process is in equilibrium. To take the example of the Ehrenfest model, there is no conflict between reversibility and the phenomenon of increasing entropy—reversibility is a property of the model in equilibrium and increasing entropy is a property of the approach to equilibrium.

If the transition rates of the Markov process $X(t)$ do satisfy the detailed balance conditions then there is an interesting alternative interpretation of the approach to equilibrium and of the function $H(t)$. In this case the forward equations (1.16) can be rewritten

$$\frac{d}{dt}u_j(t)=\sum_{k\in\mathscr{S}}\frac{1}{r(j,k)}\left(\frac{u_k(t)}{\pi(k)}-\frac{u_j(t)}{\pi(j)}\right)\qquad j\in\mathscr{S}\tag{1.20}$$

where the (possibly infinite) quantity $r(j, k)$ is given by

$$r(j,k)=[\pi(j)q(j,k)]^{-1}=[\pi(k)q(k,j)]^{-1}=r(k,j)$$

Consider now an electrical network with nodes $\mathscr{S}$ in which nodes j and k are connected by a wire of resistance $r(j, k)$ and node j is connected to earth by a capacitor of capacitance $\pi(j)$. If $u_j(t)$ is the charge present at node j at time t then $u_j(t)$, $j\in\mathscr{S}$, will satisfy equations (1.20); these are just Kirchhoff's equations and express the fact that the rate of increase of charge at node j is equal to the rate at which charge is flowing into node j. Thus the way in which probability spreads itself over the states of the Markov process is analogous to the way in which charge spreads itself over the nodes of the electrical network. Further, if we let $h(x)=-\frac{1}{2}x^2$ then

$$-H(t)=\frac{1}{2}\sum_{j\in\mathscr{S}}\frac{u_j(t)^2}{\pi(j)}$$

which is just the potential energy stored in the capacitors of the electrical network. As $H(t)$ increases, energy is dissipated as heat in the wires of the electrical network.

In this work we shall mainly be concerned with processes in equilibrium, exceptions being Section 4.5 and Chapter 5. In Chapter 5 the electrical analogue discussed here will be considered further.

Exercises 1.4

1. Show that the jump chain $X^J(t)$ of the Ehrenfest model has the same equilibrium distribution as $X(t)$. Show that if j is close to K, then in equilibrium

$$P(X^J(-1)=j-1, X^J(0)=j, X^J(1)=j-1)$$

is much larger than any of

$$P(X^J(-1)=j+1, X^J(0)=j, X^J(1)=j-1),$$
$$P(X^J(-1)=j-1, X^J(0)=j, X^J(1)=j+1),$$
$$P(X^J(-1)=j+1, X^J(0)=j, X^J(1)=j+1).$$

Deduce that if at a fixed time we observe j particles in container 1 then it is highly probable that the previous state was, and the next state will be, $j-1$.

2. If in the Ehrenfest model particles move from container 1 to container 2

at rate μ show that the equilibrium distribution is

$$\pi(j) = \left(1+\frac{\lambda}{\mu}\right)^{-K}\binom{K}{j}\left(\frac{\lambda}{\mu}\right)^j$$

3. Let $X(t)$ be a stationary stochastic process and let $\mathscr{A}$ be a subset of the state space $\mathscr{S}$. Show that

$$P(X(1), X(2), \ldots, X(n) \in \mathscr{A} \mid X(0) \in \mathscr{A})$$
$$= P(X(0), X(1), \ldots, X(n-1) \in \mathscr{A} \mid X(n) \in \mathscr{A})$$

Establish Kac's formula:

$$P(X(0) \in \mathscr{A}, X(1), X(2), \ldots, X(n) \notin \mathscr{A})$$
$$= P(X(0), X(1), \ldots, X(n-1) \notin \mathscr{A}, X(n) \in \mathscr{A})$$

Deduce that

$$P(X(1), X(2), \ldots, X(n) \notin \mathscr{A} \mid X(0) \in \mathscr{A})$$
$$= P(X(0), X(1), \ldots, X(n-1) \notin \mathscr{A} \mid X(n) \in \mathscr{A})$$

Observe that these relations hold whether the process is reversible or not.

4. Suppose the transition rates of a Markov process with a finite space satisfy the detailed balance conditions. If the process starts in state k at time $t=0$ show that

$$u_k(2t) = \sum_{j \in \mathscr{S}} \frac{\pi(k)}{\pi(j)} [u_j(t)]^2$$

Deduce from Theorem 1.6 that the function $u_k(t)$, $t \geq 0$, decreases monotonically from unity to $\pi(k)$.

1.5 KOLMOGOROV'S CRITERIA

The detailed balance conditions (1.6) enable us to decide whether a stationary Markov process is reversible from its equilibrium distribution and its transition rates. Since the equilibrium distribution is determined by the transition rates it is natural to ask whether we can establish the reversibility of a process directly from the transition rates alone. Kolmogorov's criteria allow us to do just that.

We begin by establishing the criteria for a Markov chain.

Theorem 1.7. *A stationary Markov chain is reversible if and only if its transition probabilities satisfy*

$$p(j_1, j_2)p(j_2, j_3) \cdots p(j_{n-1}, j_n)p(j_n, j_1)$$
$$= p(j_1, j_n)p(j_n, j_{n-1}) \cdots p(j_3, j_2)p(j_2, j_1) \qquad (1.21)$$

for any finite sequence of states $j_1, j_2, \ldots, j_n \in \mathscr{S}$.

Proof. If the process is reversible then the detailed balance conditions hold; hence

$$\pi(j_1)p(j_1, j_2) = \pi(j_2)p(j_2, j_1)$$
$$\pi(j_2)p(j_2, j_3) = \pi(j_3)p(j_3, j_2)$$
$$\vdots \cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pi(j_{n-1})p(j_{n-1}, j_n) = \pi(j_n)p(j_n, j_{n-1})$$
$$\pi(j_n)p(j_n, j_1) = \pi(j_1)p(j_1, j_n)$$

Multiplying these conditions together and cancelling the positive equilibrium probabilities gives equation (1.21).

Conversely, suppose the transition probabilities satisfy equation (1.21). Let j_0 be an arbitrarily chosen reference state. Since the process is irreducible, for any state $j \in \mathscr{S}$ there exists a sequence of states $j, j_n, j_{n-1}, \ldots, j_1, j_0$ leading from j to j_0 such that $p(j, j_n)p(j_n, j_{n-1}) \cdots p(j_1, j_0) > 0$. Let

$$\pi(j) = B\frac{p(j_0, j_1)p(j_1, j_2) \cdots p(j_n, j)}{p(j, j_n)p(j_n, j_{n-1}) \cdots p(j_1, j_0)}$$

where B is a positive constant. Observe that $\pi(j)$ does not depend upon the particular sequence of states chosen to lead from j to j_0, since if j, $j'_m, j'_{m-1}, \ldots, j'_1, j_0$ is another sequence of states leading from j to j_0 with $p(j, j'_m)p(j'_m, j'_{m-1}) \cdots p(j'_1, j_0) > 0$, relation (1.21) ensures that

$$\frac{p(j_0, j_1)p(j_1, j_2) \cdots p(j_n, j)}{p(j, j_n)p(j_n, j_{n-1}) \cdots p(j_1, j_0)} = \frac{p(j_0, j'_1)p(j'_1, j'_2) \cdots p(j'_m, j)}{p(j, j'_m)p(j'_m, j'_{m-1}) \cdots p(j'_1, j_0)}$$

Note that irreducibility and relation (1.21) imply that $\pi(j)$ is positive. We must now show that $\pi(j)$, $j \in \mathscr{S}$, satisfy the detailed balance conditions. If $p(j, k) = p(k, j) = 0$ these are satisfied automatically, so suppose $p(k, j) > 0$. Then we can write

$$\pi(k) = B\frac{p(j_0, j_1)p(j_1, j_2) \cdots p(j_n, j)p(j, k)}{p(k, j)p(j, j_n)p(j_n, j_{n-1}) \cdots p(j_1, j_0)}$$

Hence

$$\pi(k)p(k, j) = \pi(j)p(j, k)$$

Thus $\pi(j)$, $j \in \mathscr{S}$, satisfy the detailed balance conditions and so they also satisfy the equilibrium equations. Since the process is stationary they cannot sum to infinity; hence B can be chosen so they sum to unity. Thus from Theorem 1.2 the process is reversible and $\pi(j)$, $j \in \mathscr{S}$, is the equilibrium distribution.

Kolmogorov's criteria (1.21) provide a useful insight into the nature of a reversible Markov chain. They show that given a starting point $j_1 \in \mathscr{S}$ any path in the state space which ultimately returns to j_1 must have the same probability whether this path is traced in one direction or the other. Thus a reversible Markov chain shows no net circulation in the state space.

The proof of Theorem 1.7 has a direct analogue for a Markov process which establishes the next result.

Theorem 1.8. *A stationary Markov process is reversible if and only if its transition rates satisfy*

$$q(j_1, j_2)q(j_2, j_3) \cdots q(j_{n-1}, j_n)q(j_n, j_1)$$
$$= q(j_1, j_n)q(j_n, j_{n-1}) \cdots q(j_3, j_2)q(j_2, j_1) \qquad (1.22)$$

for any finite sequence of states $j_1, j_2, \ldots, j_n \in \mathcal{S}$.

In practice relation (1.22) does not usually have to be established for all closed paths $j_1, j_2, \ldots, j_n, j_1$ since it is often possible to choose certain simple paths so that the truth of (1.22) for a general path follows from its truth for these simple paths. For instance if relation (1.22) can be established for sequences of distinct states then it follows for all sequences. Another example is contained in Exercise 1.5.2, and a further example follows.

A two-server queue. Suppose that the stream of customers arriving at a queue forms a Poisson process of rate ν and that there are two servers who possibly differ in efficiency. Specifically, suppose that a customer's service time at server i is exponentially distributed with mean μ_i^{-1}, for $i = 1, 2$, where to ensure that equilibrium is possible $\mu_1 + \mu_2 > \nu$. If a customer arrives to find both servers free he is equally likely to be allocated to either server. The queue can be represented by a Markov process whose transition rates and associated graph G are illustrated in Fig. 1.2. State n, for $n = 0, 2, 3, \ldots$, corresponds to there being n customers in the queue, while state $1A$ or $1B$ corresponds to there being a single customer in the queue, allocated to server 1 or 2 respectively. To ensure that the process is reversible in equilibrium we need only check the relation

$$q(0, 1A)q(1A, 2)q(2, 1B)q(1B, 0) = q(0, 1B)q(1B, 2)q(2, 1A)q(1A, 0) \qquad (1.23)$$

since Kolmogorov's criterion (1.22) for any other finite sequence of states

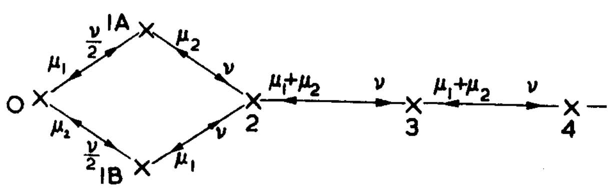

Fig. 1.2 Representation of a two-server queue

will follow from this or will hold trivially. Relation (1.23) holds, since it reduces to

$$\tfrac{1}{2}\nu \times \nu \times \mu_1 \times \mu_2 = \tfrac{1}{2}\nu \times \nu \times \mu_2 \times \mu_1$$

The equilibrium distribution is given by

$$\pi(1A) = \pi(0)\frac{\nu}{2\mu_1}$$

$$\pi(1B) = \pi(0)\frac{\nu}{2\mu_2}$$

$$\pi(n) = \pi(0)\frac{\nu^2}{2\mu_1\mu_2}\left(\frac{\nu}{\mu_1+\mu_2}\right)^{n-2} \qquad n = 2, 3, \ldots$$

Observe that if a customer arriving to find both servers free is allocated to server 1 with probability $p \neq \frac{1}{2}$ then the process is not reversible since relation (1.23) will fail to hold.

Exercises 1.5

1. There is an alternative proof of Theorem 1.7 which is instructive. By summing the equation
$$p(j, j_1)p(j_1, j_2) \cdots p(j_n, k)p(k, j) = p(j, k)p(k, j_n) \cdots p(j_2, j_1)p(j_1, j)$$
over all $j_1, j_2, \ldots, j_n \in \mathcal{S}$, and then letting $n \to \infty$, deduce that, for aperiodic chains in the first instance, the equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, satisfies
$$\pi(k)p(k, j) = \pi(j)p(j, k)$$
2. Consider a stationary Markov process with a state j_0 such that $q(j, j_0) > 0$ for all $j \in \mathcal{S}$. Show that a necessary and sufficient condition for reversibility is that
$$q(j_0, j_1)q(j_1, j_2)q(j_2, j_0) = q(j_0, j_2)q(j_2, j_1)q(j_1, j_0)$$
for all $j_1, j_2 \in \mathcal{S}$.
3. Construct a stationary Markov process which is not reversible yet which satisfies relation (1.22) when $n = 3$.
4. Consider a stationary Markov process whose associated graph G can be imbedded in the plane without any of its edges crossing. Show that the process is reversible if relation (1.22) holds for every minimal closed path, where a closed path is called minimal if there is a point in the plane such that the closed path is associated with the subgraph of G encircling the point.
5. Consider the two-server queue described in this section. Show that if $\mu_1 = \mu_2 = \mu$ then the number in the queue is a birth and death process and $\pi(0) = (2\mu + \nu)/(2\mu - \nu)$.

6. Generalize the queue described in this section to the case of s servers. Assume that if a customer arrives to find more than one server free he is equally likely to be allocated to any of them.
7. Observe that Lemma 1.5 could be regarded as a corollary of Theorem 1.8. Consider now the following amendment of the two-server queue described in this section. Suppose that if a customer arrives to find both servers free he is allocated to the server who has been free for the shortest time. Show that the resulting queue can be represented by a Markov process whose associated graph G is a tree. Generalize the queue to the case of s servers. Show that the probability servers $i_1, i_2, \ldots, i_m$ are busy and the rest free is the same as in the queue considered in the preceding exercise.

1.6 TRUNCATING REVERSIBLE PROCESSES

Various amendments can be made to the transition rates of a reversible Markov process without destroying the property of reversibility. For example if a reversible Markov process is altered by changing $q(j_1, j_2)$ to $cq(j_1, j_2)$ and $q(j_2, j_1)$ to $cq(j_2, j_1)$, where $c>0$, then the resulting Markov process is reversible and has the same equilibrium distribution. This follows from Theorem 1.3, since the detailed balance conditions (1.6) will still be satisfied. A slightly different alteration is the subject of the next lemma.

Lemma 1.9. *If the transition rates of a reversible Markov process with state space $\mathcal{S}$ and equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, are altered by changing $q(j, k)$ to $cq(j, k)$ for $j \in \mathcal{A}$, $k \in \mathcal{S}-\mathcal{A}$, where $c>0$, then the resulting Markov process is reversible in equilibrium and has equilibrium distribution*

$$\begin{aligned} &B\pi(j) && j \in \mathcal{A} \\ &Bc\pi(j) && j \in \mathcal{S}-\mathcal{A} \end{aligned}$$

where B is a normalizing constant.

Proof. The suggested equilibrium distribution satisfies the detailed balance conditions and so the result follows from Theorem 1.3. The normalizing constant is given by

$$B^{-1} = \sum_{j \in \mathcal{A}} \pi(j) + c \sum_{j \in \mathcal{S}-\mathcal{A}} \pi(j)$$

If $c=0$ the resulting process has a smaller state space. Say that a Markov process is *truncated* to the set $\mathcal{A} \subset \mathcal{S}$ if $q(j, k)$ is changed to zero for $j \in \mathcal{A}$, $k \in \mathcal{S}-\mathcal{A}$, and if the resulting process is irreducible within the state space $\mathcal{A}$. Like Lemma 1.9 the next result follows directly from the detailed balance conditions.

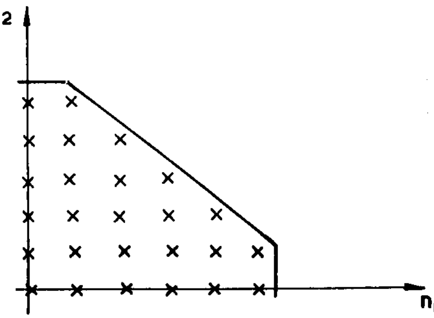

Fig. 1.3 The state space for two queues with a joint waiting room of size 4

Corollary 1.10. *If a reversible Markov process with state space $\mathcal{S}$ and equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, is truncated to the set $\mathcal{A} \subset \mathcal{S}$ then the resulting Markov process is reversible in equilibrium and has equilibrium distribution*

$$\frac{\pi(j)}{\sum_{k \in \mathcal{A}} \pi(k)} \qquad j \in \mathcal{A}$$

It is interesting to note that the equilibrium distribution of the truncated process is just the conditional probability that the original process is in state j given that it is somewhere in $\mathcal{A}$. An example has already been given in Exercise 1.3.2; another follows.

Two queues with a joint waiting room. Consider two independent $M/M/1$ queues. Let ν_i be the arrival rate and μ_i^{-1} the mean service time at queue i, for $i = 1, 2$. If n_i is the number of customers in queue i then the Markov process (n_1, n_2) is reversible (Exercise 1.2.8) with equilibrium distribution

$$\pi(n_1, n_2) = \left(1 - \frac{\nu_1}{\mu_1}\right)\left(\frac{\nu_1}{\mu_1}\right)^{n_1}\left(1 - \frac{\nu_2}{\mu_2}\right)\left(\frac{\nu_2}{\mu_2}\right)^{n_2}$$

Suppose now that the two queues are forced to share a joint waiting room of size R, so that a customer who arrives to find R customers already waiting for service, not including those being served, leaves without being served. This corresponds to truncating the Markov process (n_1, n_2) to $\mathcal{A}$, the set of states in which not more than R customers are waiting (Fig. 1.3). The equilibrium distribution for the truncated process will thus be

$$\pi(n_1, n_2) = \pi(0, 0)\left(\frac{\nu_1}{\mu_1}\right)^{n_1}\left(\frac{\nu_2}{\mu_2}\right)^{n_2} \qquad (n_1, n_2) \in \mathcal{A}$$

Exercises 1.6

1. Suppose that the two queues considered in this section have three waiting rooms associated with them: a waiting room of size R_1 for customers at

queue 1, a waiting room of size R_2 for customers at queue 2, and an overflow waiting room of size R_3 which can hold customers waiting for either queue. Identify the state space and write down the form of the equilibrium distribution.

2. Suppose that a Markov process with equilibrium distribution $\pi(j)$, $j \in \mathscr{S}$, is truncated to the set $\mathscr{A} \subset \mathscr{S}$. Show that the equilibrium distribution of the truncated process is the conditional probability distribution

$$\frac{\pi(j)}{\sum_{k \in \mathscr{A}} \pi(k)} \qquad j \in \mathscr{A}$$

if and only if the distribution $\pi(j)$, $j \in \mathscr{S}$, satisfies

$$\pi(j) \sum_{k \in \mathscr{A}} q(j, k) = \sum_{k \in \mathscr{A}} \pi(k) q(k, j) \qquad j \in \mathscr{A} \tag{1.24}$$

These equations are of a form intermediate between the detailed balance conditions (1.6) and the full balance conditions (1.3), and we shall call them the partial balance conditions for the set $\mathscr{A}$. Observe that the distribution $\pi(j)$, $j \in \mathscr{S}$, satisfies the partial balance conditions (1.24) if and only if

$$\pi(j) \sum_{k \in \mathscr{S} - \mathscr{A}} q(j, k) = \sum_{k \in \mathscr{S} - \mathscr{A}} \pi(k) q(k, j) \qquad j \in \mathscr{A}$$

These equations should be compared with equation (1.8).

3. Suppose that a Markov process with equilibrium distribution $\pi(j)$, $j \in \mathscr{S}$, is altered by changing the transition rate $q(j, k)$ to $cq(j, k)$ for $j, k \in \mathscr{A}$, where $c \neq 0$ or 1. Show that the resulting Markov process has the same equilibrium distribution if and only if the partial balance conditions (1.24) are satisfied.

4. Suppose that a Markov process with equilibrium distribution $\pi(j)$, $j \in \mathscr{S}$, is altered by changing the transition rate $q(j, k)$ to $cq(j, k)$ for $j \in \mathscr{A}$, $k \in \mathscr{S} - \mathscr{A}$, where $c \neq 0$ or 1. Show that the resulting Markov process has an equilibrium distribution of the form

$$\begin{array}{ll} B\pi(j) & j \in \mathscr{A} \\ Bc\pi(j) & j \in \mathscr{S} - \mathscr{A} \end{array}$$

if and only if the distribution $\pi(j)$, $j \in \mathscr{S}$, satisfies the partial balance conditions (1.24).

1.7 REVERSED PROCESSES

If $X(t)$ is a reversible Markov process then $X(\tau - t)$ is also a Markov process since it is statistically indistinguishable from $X(t)$. In this section we shall investigate the form of the reversed process $X(\tau - t)$ when $X(t)$ is a Markov process, but one which is not necessarily reversible.

The characterization of a Markov process as a process for which, conditional on the present, the past and the future are independent shows that if $X(t)$ is a Markov process then so is $X(\tau - t)$. An alternative proof is given in the next lemma which shows the complications that can arise if $X(t)$ is not stationary.

Lemma 1.11. *If $X(t)$ is a time homogeneous Markov process which is not stationary then the reversed process $X(\tau - t)$ is a Markov process which is not even time homogeneous.*

Proof. Since $X(t)$ is a Markov process we have the following factorization for $t_1 < t_2 < \cdots < t_n$:

$$P(j_1, j_2, \ldots, j_n) = P(j_1) \prod_{r=2}^{n} P(j_r \mid j_{r-1})$$

But

$$P(j_{r-1})P(j_r \mid j_{r-1}) = P(j_r)P(j_{r-1} \mid j_r) \tag{1.25}$$

and so

$$P(j_1, j_2, \ldots, j_n) = P(j_n) \prod_{r=2}^{n} P(j_{r-1} \mid j_r)$$

This factorization shows that $X(\tau - t)$ is Markov, but let us look more closely at the definition of $P(j_{r-1} \mid j_r)$ contained in equation (1.25). An alternative version of equation (1.25) is

$$\begin{aligned} P(X(t) = j)P(X(t+h) = k \mid X(t) = j) \\ = P(X(t+h) = k)P(X(t) = j \mid X(t+h) = k) \end{aligned} \tag{1.26}$$

Now $P(X(t+h) = k \mid X(t) = j)$ does not depend upon t, but $P(X(t) = j)$ and $P(X(t+h) = k)$ will depend upon t for some $j, k \in \mathcal{S}$ if $X(t)$ is not stationary. Thus $P(X(t) = j \mid X(t+h) = k)$ will depend upon t, and so $X(\tau - t)$ will not be time homogeneous.

If $X(t)$ is stationary the situation is much simpler.

Theorem 1.12. *If $X(t)$ is a stationary Markov process with transition rates $q(j, k)$, $j, k \in \mathcal{S}$, and equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, then the reversed process $X(\tau - t)$ is a stationary Markov process with transition rates*

$$q'(j, k) = \frac{\pi(k)q(k, j)}{\pi(j)} \qquad j, k \in \mathcal{S}$$

and the same equilibrium distribution.

Proof. From equation (1.26) we obtain

$$P(X(t)=j \mid X(t+h)=k)=\frac{\pi(j)}{\pi(k)}P(X(t+h)=k \mid X(t)=j)$$

Now divide both sides by h and let h tend to zero. Thus

$$q'(k, j)=\frac{\pi(j)q(j, k)}{\pi(k)}$$

The fact that the reversed process is stationary follows as an immediate consequence of the definition of stationarity. That $X(t)$ and $X(\tau-t)$ have the same equilibrium distribution follows since they have the same stationary distribution, but it is worth checking that the equilibrium equations

$$\pi(j)\sum_{k\in\mathcal{S}} q'(j, k)=\sum_{k\in\mathcal{S}} \pi(k)q'(k, j)$$

are satisfied.

The next example illustrates the theorem.

A two-server queue. Suppose the stream of customers arriving at a two-server queue forms a Poisson process of rate ν and that a customer's service time at server i is exponentially distributed with mean μ_i^{-1}, for $i=1, 2$, where $\mu_1+\mu_2>\nu$. If a customer arrives to find both servers free he is allocated to the server who has been free for the longest time. The queue can be represented by a Markov process whose transition rates and associated graph G are illustrated in Fig. 1.4(a). State n, for $n=2, 3, \ldots,$

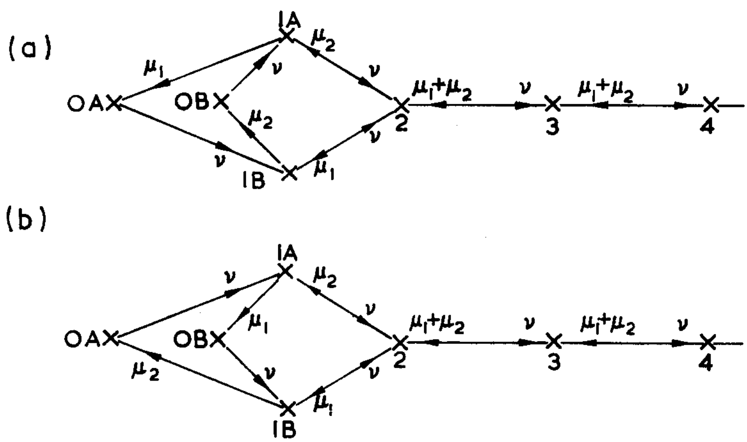

Fig. 1.4 A two-server queue: (a) the original process and (b) the reversed process

corresponds to there being n customers in the queue. State 1A or 1B corresponds to there being a single customer in the queue, allocated to server 1 or 2 respectively. State 0A or 0B corresponds to both servers being free, with server 1 or 2 respectively having been free for the shorter time. The process is clearly not reversible since $q(0A, 1B)$ is positive and $q(1B, 0A)$ is zero. The equilibrium distribution for the process is

$$\pi(n) = \pi(2)\left(\frac{\nu}{\mu_1 + \mu_2}\right)^{n-2} \qquad n = 2, 3, \ldots$$

$$\pi(1A) = \pi(2)\frac{\mu_2}{\nu}$$

$$\pi(1B) = \pi(2)\frac{\mu_1}{\nu}$$

$$\pi(0A) = \pi(0B) = \pi(2)\frac{\mu_1\mu_2}{\nu^2}$$

Theorem 1.12 shows that the transition rates of the reversed process are as illustrated in Fig. 1.4(b). Observe that they take a particularly simple form. This is not always the case, as Exercise 1.7.1 demonstrates.

Remember that the period for which $X(t)$ remains in state j is exponentially distributed with parameter

$$q(j) = \sum_{k \in \mathscr{S}} q(j, k)$$

Similarly, define

$$q'(j) = \sum_{k \in \mathscr{S}} q'(j, k)$$

It follows from Theorem 1.12 that $q(j) = q'(j)$. This is not surprising: the periods spent in state j have the same distribution whatever the direction of time. Theorem 1.12 has the following converse.

Theorem 1.13. *Let $X(t)$ be a stationary Markov process with transition rates $q(j, k)$, $j, k \in \mathscr{S}$. If we can find a collection of numbers $q'(j, k)$, $j, k \in \mathscr{S}$, such that*

$$q'(j) = q(j) \qquad j \in \mathscr{S} \tag{1.27}$$

and a collection of positive numbers $\pi(j)$, $j \in \mathscr{S}$, summing to unity, such that

$$\pi(j)q(j, k) = \pi(k)q'(k, j) \qquad j, k \in \mathscr{S} \tag{1.28}$$

then $q'(j, k)$, $j, k \in \mathscr{S}$, are the transition rates of the reversed process $X(\tau - t)$ and $\pi(j)$, $j \in \mathscr{S}$, is the equilibrium distribution of both processes.

Proof. From equations (1.28) and (1.27) it follows that

$$\sum_{j \in \mathcal{S}} \pi(j)q(j, k) = \pi(k) \sum_{j \in \mathcal{S}} q'(k, j)$$
$$= \pi(k)q'(k)$$
$$= \pi(k)q(k)$$

Thus $\pi(j)$, $j \in \mathcal{S}$, is the equilibrium distribution of $X(t)$. That $q'(j, k)$, $j, k \in \mathcal{S}$, are the transition rates of the reversed process then follows from Theorem 1.12.

We shall find Theorem 1.13 useful in Chapter 3 where we discuss a rather complicated Markov process for which it would be tedious to check the equilibrium equations, but for which possible transition rates of the reversed process are apparent. The similarity of equation (1.28) to the detailed balance condition should be observed. A generalization of Kolmogorov's criteria can also be obtained (Exercise 1.7.4).

Occasionally we may come across a stationary Markov process for which the reversed process, while not statistically indistinguishable from the original process, would be if some of the states were interchanged. To make this notion precise suppose that to each state $j \in \mathcal{S}$ there corresponds a conjugate state $j^+ \in \mathcal{S}$ with $(j^+)^+ = j$. Then the stationary Markov process $X(t)$ is called *dynamically reversible* if $X(t)$ is statistically indistinguishable from $[X(\tau - t)]^+$. As an example consider the stationary Markov process with state space $\mathcal{S} = \{-n, -n+1, \ldots, n-1, n\}$ and transition rates

$$q(j, j+1) = \lambda \qquad j = -n, -n+1, \ldots, n-1$$
$$q(n, -n) = \lambda$$

With $j^+ = -j$ this process is dynamically reversible. Reversing this process has an analogous effect to reversing the velocity of a particle moving in a circular orbit—hence the term 'dynamically reversible'. A further example is the two-server queue illustrated in Fig. 1.4, which is dynamically reversible with $(0A)^+ = 0B$ and all other states self-conjugate.

Theorem 1.14. *A stationary Markov process with $q(j) = q(j^+)$, $j \in \mathcal{S}$, is dynamically reversible if and only if there exists a collection of positive numbers $\pi(j)$, $j \in \mathcal{S}$, summing to unity that satisfy*

$$\pi(j) = \pi(j^+) \qquad j \in \mathcal{S} \qquad (1.29)$$

and

$$\pi(j)q(j, k) = \pi(k^+)q(k^+, j^+) \qquad j, k \in \mathcal{S}$$

When there exists such a collection $\pi(j)$, $j \in \mathcal{S}$, it is the equilibrium distribution of the process.

Proof. If the process is dynamically reversible then the relations follow from the identification $\pi(j) = P(X(t) = j)$. Conversely, if the process satisfies the relations let $q'(k, j) = q(k^+, j^+)$. Thus

$$q'(k, j) = \frac{\pi(j)}{\pi(k^+)} q(j, k)$$
$$= \frac{\pi(j)}{\pi(k)} q(j, k)$$

Further,

$$q'(j) = \sum_{k \in \mathcal{S}} q'(j, k)$$
$$= \sum_{k \in \mathcal{S}} q(j^+, k^+)$$
$$= q(j^+)$$
$$= q(j)$$

We have thus established that the transition rates $q'(j, k)$, $j, k \in \mathcal{S}$, satisfy equations (1.27) and (1.28) and so, by Theorem 1.13, $\pi(j)$, $j \in \mathcal{S}$, is the equilibrium distribution and the reversed process $X(\tau - t)$ has transition rates $q'(j, k)$, $j, k \in \mathcal{S}$. Since $q'(j, k) = q(j^+, k^+)$ the process $X(t)$ is dynamically reversible.

Exercises 1.7

1. If $X(t)$ is the stationary Markov process whose transition rates were given in Exercise 1.1.4, with $a \leq 1$ and $b > 0$, find the transition rates of the reversed process $X(\tau - t)$.
2. Construct examples to show that condition (1.27) cannot be dropped from Theorem 1.13, nor condition (1.29) from Theorem 1.14.
3. Establish counterparts of Theorems 1.12, 1.13, and 1.14 for Markov chains. Observe that no analogue of condition (1.27) is needed: the implicit condition that transition probabilities sum to unity serves the same purpose.
4. Let $X(t)$ be a stationary Markov chain with transition probabilities $p(j, k)$, $j, k \in \mathcal{S}$. Show that if there exist transition probabilities $p'(j, k)$, $j, k \in \mathcal{S}$, such that

   $$p(j_1, j_2)p(j_2, j_3) \cdots p(j_{n-1}, j_n)p(j_n, j_1) = p'(j_1, j_n)p'(j_n, j_{n-1}) \cdots p'(j_3, j_2)p'(j_2, j_1)$$

   for any finite sequence of states $j_1, j_2, \ldots, j_n \in \mathcal{S}$, then $p'(j, k)$, $j, k \in \mathcal{S}$, are the transition probabilities of the reversed Markov chain $X(\tau - t)$. Using the additional condition (1.27) obtain the parallel result for a Markov process.

5. Show that the reversed process illustrated in Fig. 1.4(b) can be regarded as representing a two-server queue identical to the one represented by the original process but with states 0A or 0B indicating that the next arrival will be allocated to server 1 or 2 respectively.
6. Generalize the queue considered in this section to the case of s servers. Show that the probability servers $i_1, i_2, \ldots, i_m$ are busy and the rest free is the same as in the queues considered in Exercises 1.5.6 and 1.5.7.
7. Suppose that a Markov process $X(t)$ with transition rates $q(j, k)$, $j, k \in \mathcal{S}$, and equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$, is truncated to the set $\mathcal{A}$. Let $Y(t)$ be the stationary truncated process. Let $Z(t)$ be the stationary process resulting from truncating the reversed process $X(-t)$ to the set $\mathcal{A}$. Show that $Z(t)$ and $Y(-t)$ have the same transition rates if and only if the partial balance conditions (1.24) are satisfied. If $Z(t)$ and $Y(-t)$ have the same transition rates we shall say that for the process $X(t)$ the operations of time reversal and truncation to the set $\mathcal{A}$ commute.
8. Consider a Markov process with transition rates $q(j, k)$, $j, k \in \mathcal{S}$, and equilibrium distribution $\pi(j)$, $j \in \mathcal{S}$. Suppose that the probability flux out of the set $\mathcal{A}$

$$\sum_{j \in \mathcal{A}} \sum_{k \in \mathcal{S} - \mathcal{A}} \pi(j) q(j, k)$$

is finite. Show that the Markov chain formed by observing the process at those instants in time just before it leaves the set $\mathcal{A}$ has the same equilibrium distribution as the Markov chain formed by observing the process at those instants in time just after it enters the set $\mathcal{A}$ if and only if the partial balance conditions (1.24) are satisfied.